Approximations of spindle convex sets

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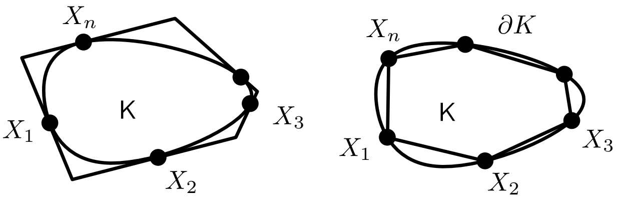
The results presented in this talk are joint with Viktor Vígh.

Approximations of ovals by (linear) polygons

- An oval K is a compact, convex sets in the plane with nonempty interior.
- We approximate an oval K by inscribed and circumscribed (linear) polygons with a given number of vertices/edges.



Distances used in this talk

The goodness of the approximation can be measured by various notions of distance.

- $K_1, K_2 \subset \mathbb{R}^2$ ovals containing the origin o.
- h_1 and h_2 denote the support functions of K_1 and K_2, respectively.
- We use δ_H, δ_A and δ_ℓ to denote the Hausdorff-metric, area-deviation, and perimeter-deviation, respectively. More precisely,

$$\delta_H(K_1, K_2) = \max_{\varphi \in [0, 2\pi)} |h_1(\varphi) - h_2(\varphi)|,$$

$$\delta_A(K_1, K_2) = A(K_1 \backslash K_2) + A(K_2 \backslash K_1),$$

$$\delta_\ell(K_1, K_2) = \left| \int_0^{2\pi} h_1(\varphi) - h_2(\varphi) d\varphi \right|.$$

Best approximating (linear) polygons

- Let P_n^H, P_n^A, and P_n^ℓ ($P_{(n)}^H$, $P_{(n)}^A$ and $P_{(n)}^\ell$) denote (linear) polygons with at most n sides inscribed in K (circumscribed about K) that are closest to K with respect to the Hausdorff-metric, area-deviation, and perimeter-deviation, respectively.
- Such a (not necessarily unique) minimizer exists for each one of the three measures of distance in each case.
- In each one of the six cases the distance of the minimizer and K approaches zero as n tends to infinity.

Theorem (McClure and Vitale (1975))

Let K be an oval with C^2 boundary. Then the following hold as $n \to \infty$.

$$\delta_\ell(K, P_n^\ell) \sim \frac{1}{24}\left(\int_{\partial K} \kappa(s)^{\frac{2}{3}}\, ds\right)^3 \cdot \frac{1}{n^2}, \tag{i}$$

$$\delta_A(K, P_n^A) \sim \frac{1}{12}\left(\int_{\partial K} \kappa(s)^{\frac{1}{3}}\, ds\right)^3 \cdot \frac{1}{n^2}, \tag{ii}$$

$$\delta_H(K, P_n^H) \sim \frac{1}{8}\left(\int_{\partial K} \kappa(s)^{\frac{1}{2}}\, ds\right)^2 \cdot \frac{1}{n^2}, \tag{iii}$$

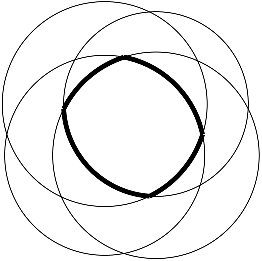
$$\delta_\ell(K, P_{(n)}^\ell) \sim \frac{1}{12}\left(\int_{\partial K} \kappa(s)^{\frac{2}{3}}\, ds\right)^3 \cdot \frac{1}{n^2}, \tag{iv}$$

$$\delta_A(K, P_{(n)}^A) \sim \frac{1}{24}\left(\int_{\partial K} \kappa(s)^{\frac{1}{3}}\, ds\right)^3 \cdot \frac{1}{n^2}, \tag{v}$$

$$\delta_H(K, P_{(n)}^H) \sim \frac{1}{8}\left(\int_{\partial K} \kappa(s)^{\frac{1}{2}}\, ds\right)^2 \cdot \frac{1}{n^2}. \tag{vi}$$

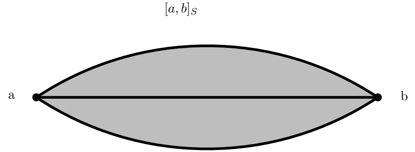
(Convex) Disc-polygons

- A disc-polygon is the intersection of a finite family of equal radius circular disc in the Euclidean plane. We usually assume that the common radius of the discs is one.



Closed spindle

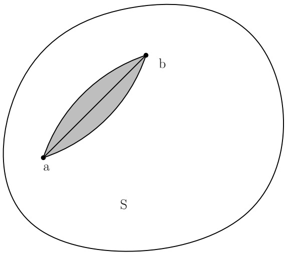
- Let $a, b \in \mathbb{E}^2$. If $d(a, b) < 2$ then the closed spindle $[a, b]_S$ of a and b is the union of the closed segment $[a, b]$ and of those circular arcs connecting a and b which have radii at least 1 and are shorter than π.



- If $d(a, b) = 2$ then $[a, b]_S$ is the unit disc with center $(a + b)/2$, otherwise $[a, b]_S$ is defined to be the whole $\mathbb{E}^d$.

Spindle convexity (Mayer, 1935)

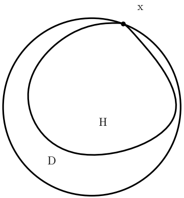
- A set $S \subseteq \mathbb{R}^2$ is called spindle convex if from $a, b \in S$ it follows that $[a, b]_S \subseteq S$.



- For example, disc-polygons are spindle convex sets.

Supporting discs

- Let $x \in \partial S$, then there exists a unit disc D such that $S \subseteq D$ and $x \in \partial D$. Such a disc is called a supporting disc at x.



- The following were proved by Bezdek,Lángi, Naszódi and Papez (see reference in next slide).
- Every closed spindle convex set is the intersection of its supporting discs.
- Closed spindle convex sets are exactly those sets that are intersections of families (not necessarly finite) of unit circular discs.

Further information on spindle convex sets

- K. Bezdek, Z. Lángi, M. Naszódi and P. Papez, *Ball-Polyhedra*, Discrete and Computational Geometry, **38** (2007), no. 2, 201–230.

Approximations of spindle convex sets by dics-polygons

- Let $S \subset \mathbb{E}^2$ be a compact spindle convex set with C^2 boundary. We consider approximations of S by inscribed and circumscribed disc-polygons with at most n sides with respect to the Hausdorff-metric, area-deviation, and perimeter-deviation.
- Let S_n^H, S_n^A, and S_n^ℓ ($S_{(n)}^H, S_{(n)}^A$ and $S_{(n)}^\ell$) denote disc-polygons with at most n sides inscribed in S (circumscribed about S) that are closest to S with respect to the Hausdorff-metric, area-deviation, and perimeter-deviation, respectively.
- Such (not necessarily unique) minimizers clearly exist for each one of the three measures of distance.
- In each one of the six cases the distance of the minimiser and S approaches zero as n tends to infinity. The main results of this talk are the following sharp estimates of the order of convergence of the distances of the minimizers to S as n tends to infinity.

Theorem (F.F., V. Vígh, 2011)

Let S be a compact, spindle convex set with C^2 boundary. Then the following hold as $n \to \infty$.

$$\delta_\ell(S, S_n^\ell) \sim \frac{1}{24}\left(\int_{\partial S}(\kappa^2(s)-1)^{\frac{1}{3}}\,ds\right)^3 \cdot \frac{1}{n^2}, \qquad \text{(i)}$$

$$\delta_A(S, S_n^A) \sim \frac{1}{12}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{3}}\,ds\right)^3 \cdot \frac{1}{n^2}, \qquad \text{(ii)}$$

$$\delta_H(S, S_n^H) \sim \frac{1}{8}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{2}}\,ds\right)^2 \cdot \frac{1}{n^2}, \qquad \text{(iii)}$$

$$\delta_\ell(S, S_{(n)}^\ell) \sim \frac{1}{24}\left(\int_{\partial S}(2\kappa^2(s)-3\kappa(s)+1)^{\frac{1}{3}}\,ds\right)^3 \cdot \frac{1}{n^2}, \qquad \text{(iv)}$$

$$\delta_A(S, S_{(n)}^A) \sim \frac{1}{24}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{3}}\,ds\right)^3 \cdot \frac{1}{n^2}, \qquad \text{(v)}$$

$$\delta_H(S, S_{(n)}^H) \sim \frac{1}{8}\left(\int_{\partial S}(\kappa(s)-1)^{\frac{1}{2}}\,ds\right)^2 \cdot \frac{1}{n^2}. \qquad \text{(vi)}$$

- We will look at the proof of only (i).
- (ii), (iv) and (v) are proved in a similar manner.
- The proofs of (iii) and (vi) follow a different line of argument and they are significantly more complicated than the proofs of the other statements.

- We use the following result by McClure (1975), and McClure and Vitale (1975).
- Let $f : [a, b] \to \mathbb{R}$ and let T_n denote a partition of $[a, b]$ of the form $T_n = (t_0, t_1, \ldots, t_n)$, where

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b.$$

- Consider a functional $E(f, T_n)$ that admits a decomposition of the following additive form, relative to T_n:

$$E(f, T_n) = \sum_{i=0}^{n-1} e(f, t_i, t_{i+1}).$$

Furthermore, let $E_n(f) = \inf_{T_n} E(f, T_n)$.

The following three assumptions will imply a theorem we will use.

Assumption (I)

For any (α, β) *satisfying* $a \leq \alpha < \beta \leq b$, $e(f, \alpha, \beta) \geq 0$*. Further,* $e(f, \cdot, \cdot)$ *is subadditive over contigous subintervals of* $[a, b]$*, that is, if* $a \leq \alpha < \beta < \gamma \leq b$ *then*

$$e(f, \alpha, \beta) + e(f, \beta, \gamma) \leq e(f, \alpha, \gamma).$$

Assumption (II)

$e(f, \alpha, \beta)$ *depends continuously on* (α, β)*.*

Assumption (III)

There is a function J_f on $[a, b]$, associated with f, and a constant $m > 1$ such that

(i) *J_f is nonnegative and piecewise continuous on $[a, b]$, admitting at worst a finite number of jump discontinuities, and*

(ii)

$$\lim_{h \to 0+} \frac{e(f, \alpha, \alpha + h)}{h^m} = J_f(\alpha+).$$

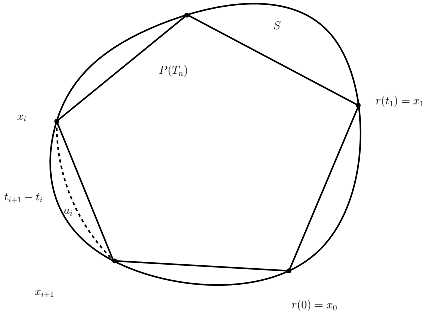
This limit is uniform in that the difference $|J_f(\alpha+) - e(f, \alpha, \alpha + h)/h^m|$ can be made uniformly small when $(\alpha, \alpha + h)$is contained in an interval where J_f is continuous.

Theorem (McClure, Vitale (1975))

If Assumptions I-III hold for $e(f, \alpha, \beta)$, then

$$\lim_{n \to \infty} n^{m-1} E_n(f) = \left(\int_a^b (J_f(s))^{1/m} ds \right)^m.$$

- Let $S \subset \mathbb{E}^2$ be a compact, spindle convex set, and let L denote its perimeter. Let $r : [0, L] \to \mathbb{R}^2$ be the arc-length parametrization of the boundary ∂S with a fixed $r(0) = x_0 \in \partial S$ starting point such that $r(s)$ defines the positive orientation of ∂S.
- Let $f : [0, L] \to \mathbb{R}$ be defined as $f(s) = s$.
- We associate with a partition $T_n = (t_0, t_1, \ldots, t_n)$ of the interval $[0, L]$ the disc-polygon $P(T_n)$ with vertex set $\{x_0, x_1, \ldots, x_{n-1}\}$ such that the arc joining x_0 with x_i is of length t_i.
- We use a_i to denote the length of the shorter unit circular arc joining x_i and x_{i+1}.



- Let $E(T_n) = \delta_\ell(P(T_n), S)$ which is simply the perimeter deviation of S and $P(T_n)$. Clearly

$$E(T_n) = \sum_{i=0}^{n-1} e(t_i, t_{i+1}),$$

where $e(t_i, t_{i+1}) = t_{i+1} - t_i - a_i$.

- Assumptions I and II are clearly satisfied for our specific f, e and E.

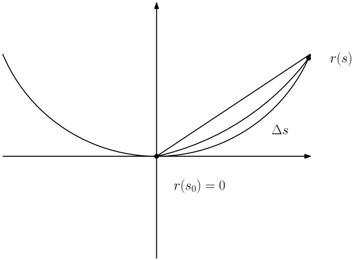
- We will prove that $J(t) = (\kappa^2(t) - 1)/24$ with $m = 3$ satisfies Assumption III. Here $\kappa(t)$ is the curvature of the boundary of S at $r(t)$.
- If we can prove this, then the above theorem of McClure and Vitale readily implies (i).

Lemma

Let K be a convex disc with C^2 boundary. Let $r(s) : [0, L] \to \mathbb{R}^2$ be the arc-length parametrization of ∂K. Let $r(s_0) = x \in \partial K$ be a fixed point. Then

$$\lim_{\Delta s \to 0} \frac{\Delta s - d(r(s_0 + \Delta s), r(s_0))}{(\Delta s)^3} = \frac{\kappa(s_0)^2}{24}.$$

- Now we need to calculate the limit of the difference between the arclenthg $s - s_0$ and length of the short circular arc joining $r(s)$ and $r(s_0)$.



In order to check part (ii) of Assumption III, note that $d(r(s_0 + \Delta s), r(s_0)) < a(s_0 + \Delta s, s_0) \leq \Delta s$ implies

$$\lim_{\Delta s \to 0^+} \frac{a(s_0 + \Delta s, s_0)}{\Delta s} = 1. \quad (1)$$

Now,

$$\lim_{\Delta s \to 0^+} \frac{e(s_0, s_0 + \Delta s)}{(\Delta s)^3} = \lim_{\Delta s \to 0^+} \frac{\Delta s - a(s_0, s_0 + \Delta s)}{(\Delta s)^3}$$
$$= \lim_{\Delta s \to 0^+} \frac{[\Delta s - d(r(s_0 + \Delta s), r(s_0))]}{(\Delta s)^3} -$$
$$- \lim_{\Delta s \to 0^+} \frac{[a(s_0 + \Delta s, s_0) - d(r(s_0 + \Delta s, r(s_0))]}{(\Delta s)^3}$$
$$= \frac{\kappa^2(s_0)}{24} - \frac{1}{24} = J(s_0).$$

Thus, McClure and Vitale's theorem yields

$$\lim_{n \to \infty} n^2 E_n(f) = \left(\int_0^L J(s)^{1/3} ds \right)^3 = \frac{1}{24} \left(\int_{\partial S} (\kappa(s)^2 - 1)^{1/3} ds \right)^3.$$

- The proof of part (ii) of the main theorem is very similar to that of part (i).
- The proofs of parts (iii) and (vi) stand apart from those of (i), (ii), (iv) and (v) in the sense that we need to use a different setup and a slightly different version of the theorem of McClure and Vitale from their 1975 paper.
- In the proofs of parts (iii)-(vi), the general idea of the argument is that we replace the boundary of S locally by its osculating circle and perfom similar computations for this osculating circle as seen in the proof of part (i). Next we show that the osculating circle is close enough to the original curve (∂S) in the sense that using it instead of ∂S does not change the limits involved.
- In the circumscribed cases (iii-vi) these ideas need to be combined with various technical estimates.

R-spindle convexity

- For any $R > 0$, one can define convex R-disc-polygons as the intersections of a finite number of closed circular discs of radius R in $\mathbb{E}^2$.
- Similarly, one can define R-spindle convexity using circles of radius R.
- We can easily rephrase the results of the main theorem for the case of best approximations of R-spindle convex planar sets by convex R-disc-polygons.
- Naturally, in this case the formulas in the main theorem will involve the radius R. The proof of the theorem remains essentially the same.

R-spindle convexity

- Assume that K is a convex disc in $\mathbb{E}^2$ with a C^2 boundary. If the minimum of the curvature of ∂K is positive, say $\kappa_{\min}$, then it can be shown that K is R-spindle convex for all $R \geq 1/\kappa_{\min}$.
- Therefore K can be approximated by inscribed and circumscribed R-disc-polygons and the order of approximation is described by the suitably modified version of then main theorem.
- If $R \to \infty$, then the integrals in the formulas in (the modified version of) the main theorem all converge to the corresponding integrals in the formulas proved by McClure and Vitale. It is not hard to see, by taking the limit $R \to \infty$, that the modified main theorem directly implies the corresponding results of McClure and Vitale for convex discs with C^2_+ boundary.

The distribution of vertices

Define the empirical distribution of the vertices of S_n^ℓ on the interval $[0, L]$ as follows:

$$F_n(x) = \frac{|\{v \in \mathrm{vert} S_n^\ell \,|\, v = r(t) \text{ with } t < x\}|}{n}.$$

It also follows by McClure and Vitale that the sequence of empirical distributions $\{F_n\}$ coverges pointwise to the distribution

$$F(x) = \frac{\int_0^x (\kappa^2(s) - 1)^{1/3} ds}{\int_{\partial S} (\kappa^2(s) - 1)^{1/3} ds}.$$

The distribution of vertices

The converse is also true in the following sense: if we define the sequence of disc-poligons $\{D_n\}$ such that for every n the vertices of D_n form a uniform partition of ∂S with respect to the density $(\kappa^2(s) - 1)^{1/3}$, then $\{D_n\}$ approximates S asymptotically efficiently, that is

$$\delta_\ell(S, D_n) \sim \frac{1}{24}\left(\int_{\partial S}(\kappa^2(s) - 1)^{1/3} ds\right)^3 \cdot \frac{1}{n^2}.$$

Conjecture (work in progress): Choose n independent, random points on ∂S with respect to the distribution F, and let the spindle convex hull be R_n. The sequence $\{R_n\}$ approximates S asymptotically effectively with probability 1.

Thank you for your attention.