Carathéodory-type Results for Faces of Convex Sets

Valeriu Soltan
George Mason University, Virginia
(joint research with Jim Lawrence)

A well-known result of convex geometry states that any point z of a compact convex set $K \subset \mathbb{R}^n$ can be expressed as a convex combination of $n + 1$ or fewer extreme points of K. Similarly, if K is a line-free closed convex set in $\mathbb{R}^n$, then z is a convex combination of $n+1$ or fewer points such that each of the points is either extreme or belongs to an extreme ray of K (Klee, 1957).

If it is desirable to express z as a convex combination of a smaller than $n+1$ number of points from the boundary of K, then, instead of extreme points or rays, one can consider faces of K. Our goal here is to study such extreme representations combined with the operations of addition and union of convex sets.

We recall that an (extreme) *face* of a convex set $K \subset \mathbb{R}^n$ is a convex subset $F \subset K$ such that points $x, y \in K$ lie in F provided $(1-\lambda)x + \lambda y \in F$ for a suitable scalar $0 < \lambda < 1$.

1. *Let $K_1, \dots, K_r$ be nonempty line-free closed convex sets in $\mathbb{R}^n$. For any point $z \in K_1 + \cdots + K_r$, there are nonempty faces F_i of K_i, $i = 1, \dots, r$, such that $z \in F_1 + \cdots + F_r$ and*

$$\dim F_1 + \cdots + \dim F_r \leq n.$$

Sketch of the Proof. Choose exposed points v_i of K_i, $i = 1, \dots, r$, and put

$$\bar{v} = (v_1, \dots, v_r), \quad K = K_1 \times \cdots \times K_r.$$

One can show the existence of a nonsingular linear transformation f on $(\mathbb{R}^n)^r$ such that $f(\bar{v})$ is the unique lexicographically minimal point of $f(K)$. Let

$$L = \{\bar{x} = (x_1, \dots, x_r) \in (\mathbb{R}^n)^r \mid x_1 + \cdots + x_r = o\}.$$

For any point $\bar{x} \in K$, denote by $\varphi(\bar{x})$ the inverse image of the unique lexicographically minimal point of the set $f(K \cap (\bar{x} + L))$.

Put $B = \varphi(K)$. Then $B = \{\bar{x} \in K \mid \varphi(\bar{x}) = \bar{x}\}$.

If $z \in K_1 + \cdots + K_r$, then $z = z_1 + \cdots + z_r$ for suitable points $z_i \in K_i$, $i = 1, \ldots, r$. Put

$$\bar{z} = (z_1, \ldots, z_r) \quad \text{and} \quad \varphi(\bar{z}) = (z_1', \ldots, z_r').$$

Since $\varphi(\bar{z}) \in K \cap (\bar{z} + L)$, we have $\varphi(\bar{z}) = \bar{z} + \bar{x}$ for some point $\bar{x} \in L$. Hence

$$z_1' + \cdots + z_r' = (z_1 + \cdots + z_r) + (x_1 + \cdots + x_r) = z.$$

Denote by F the face of K that contains $\varphi(\bar{z})$ in its relative interior. It is possible to show that $F \subset B$.

We can write $F = F_1 \times \cdots \times F_s$, where F_i is a nonempty face of K_i, $i = 1, \ldots, r$. From $\varphi(\bar{z}) \in F$ it follows that $z_i' \in F_i$ for all $i = 1, \ldots, r$. Since the linear transformation $g : (\mathbb{R}^n)^r \to \mathbb{R}^n$, defined by

$$g(x_1, \ldots, x_r) = x_1 + \cdots + x_r,$$

is one-to-one on B, one has

$$\dim F_1 + \cdots + \dim F_r = \dim F = \dim g(F) \leq n.$$

Finally,

$$z = z_1' + \cdots + z_r' \in F_1 + \cdots + F_r.$$

If the number r above is greater than n, then at least $r - n$ of the faces F_i are singletons. This argument enables the refinement of the Shapley-Folkman lemma (Starr, 1969): For any compact sets $X_1, \ldots, X_r \subset \mathbb{R}^n$ and a point $z \in \operatorname{conv}(X_1 + \cdots + X_r)$, there is an index set $I \subset \{1, \ldots, r\}$ with $|I| \leq n$ such that

$$z \in \sum_{i \in I} \operatorname{conv} X_i + \sum_{i \notin I} X_i.$$

2. *For any sets* $X_1, \ldots, X_r \subset \mathbb{R}^n$ *and a point* z *in* $\operatorname{conv}(X_1 + \cdots + X_r)$, *there is an index set* $I \subset \{1, \ldots, r\}$ *with* $|I| \leq n$ *and subsets* $Y_i \subset X_i$ *such that*

$$z \in \sum_{i \in I} \operatorname{conv} Y_i + \sum_{i \notin I} Y_i, \quad \sum_{i \in I} |Y_i| \leq n + |I|.$$

and $|Y_i| = 1$ *for all* $i \notin I$.

Our next result deals with unions of convex sets.

3. *Let* $K_1, \ldots, K_r \subset \mathbb{R}^n$ *be line-free closed convex sets. For any point* $z \in \operatorname{conv}(K_1 \cup \cdots \cup K_r)$*, there is an index set*

$$I \subset \{1, \ldots, r\} \quad \text{with} \quad |I| \leq n+1$$

and faces F_i *of* K_i, $i \in I$, *such that*

$$z \in \operatorname{conv}\big(\bigcup_{i \in I} F_i\big) \quad \text{and} \quad \sum_{i \in I} \dim F_i \leq n. \tag{1}$$

If, additionally, all $K_1, \ldots, K_r$ *are compact, then the inequality in* (1) *can be refined as*

$$\sum_{i \in I} \dim F_i \leq n + 1 - |I|.$$

From **3**, we obtain the following corollary.

4. *Let $K \subset \mathbb{R}^n$ be a line-free closed convex set and r a positive integer. For any point $z \in K$, there are faces $F_1, \dots, F_s$ of K, where $s \leq \min\{r, n+1\}$, such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$$

and

$$\dim F_1 + \cdots + \dim F_s \leq n. \tag{2}$$

If $r > 1$, then $F_1, \dots, F_s$ can be chosen proper in K such that at least $s-1$ of them are of dimension one or less.

If K is compact, then the inequality (2) can be refined as

$$\dim F_1 + \cdots + \dim F_s \leq n + 1 - s.$$

The paper

È. A. Danielyan, G. S. Movsisyan, K. R. Tatalyan, *Generalization of the Carathéodory theorem,* (Russian) Akad. Nauk Armenii Dokl. **92** (1991), 69–75,

deals with a sharper version of **4**, formulated by us as a problem.

Problem. *Let* $K \subset \mathbb{R}^n$ *be a compact convex set and* $n_1, \ldots, n_s$ *positive integers with* $n_1 + \cdots + n_s = n + 1$. *Prove that for any point* $z \in K$, *there are nonempty faces* $F_1, \ldots, F_s$ *of* K *such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$$

and

$$\dim F_i \leq n_i - 1 \quad \textit{for all} \quad i = 1, \ldots, s.$$

5. *The Problem above has an affirmative answer when* K *is a convex polytope.*

One more result deals with intersections of convex polytopes in $\mathbb{R}^n$.

6. *Let* $P_1, \ldots, P_s \subset \mathbb{R}^n$ *be polytopes and* $n_1, \ldots, n_s$ *positive integers with* $n_1 + \cdots + n_s = n + 1$. *For any point* $z \in P_1 \cap \cdots \cap P_s$, *there are nonempty faces* F_i *of* P_i, $i = 1, \ldots, s$, *such that*

$$z \in \operatorname{conv}(F_1 \cup \cdots \cup F_s)$$

and

$$\dim F_i \leq n_i - 1 \quad \textit{for all} \quad i = 1, \ldots, s.$$

With $s = n+1$ and $n_i = 1$, $i = 1, \ldots, n+1$, **6** gives a new way to prove "the colorful version" of Carathéodory's theorem due to Bárány (1982), which states that, given nonempty sets $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ and a point $z \in \operatorname{conv} X_1 \cap \cdots \cap \operatorname{conv} X_{n+1}$, there are points $v_i \in X_i$, $i = 1, \ldots, n+1$, such that $z \in \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$.