Convex Sets in Empty Convex Position

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Let $f(n)$ be the smallest number such that any set of $f(n)$ points in general position in the plane has a subset of n points which are in convex position.

It is conjectured that

$$f(n) = 2^{n-2} + 1.$$

The best upper bound for $f(n)$ is due to Géza Tóth and Pavel Valtr:

$$f(n) \leq \binom{2n-5}{n-2} + 1.$$

Let S be a set of points in general position in the plane. A subset of n points of S is an *empty convex n-gon*, if they form the vertices of a convex n-gon which does not contain any point of S in its interior. Erdős asked whether for any n a sufficiently large cardinality of S guaranties the existence of an empty n-gon.

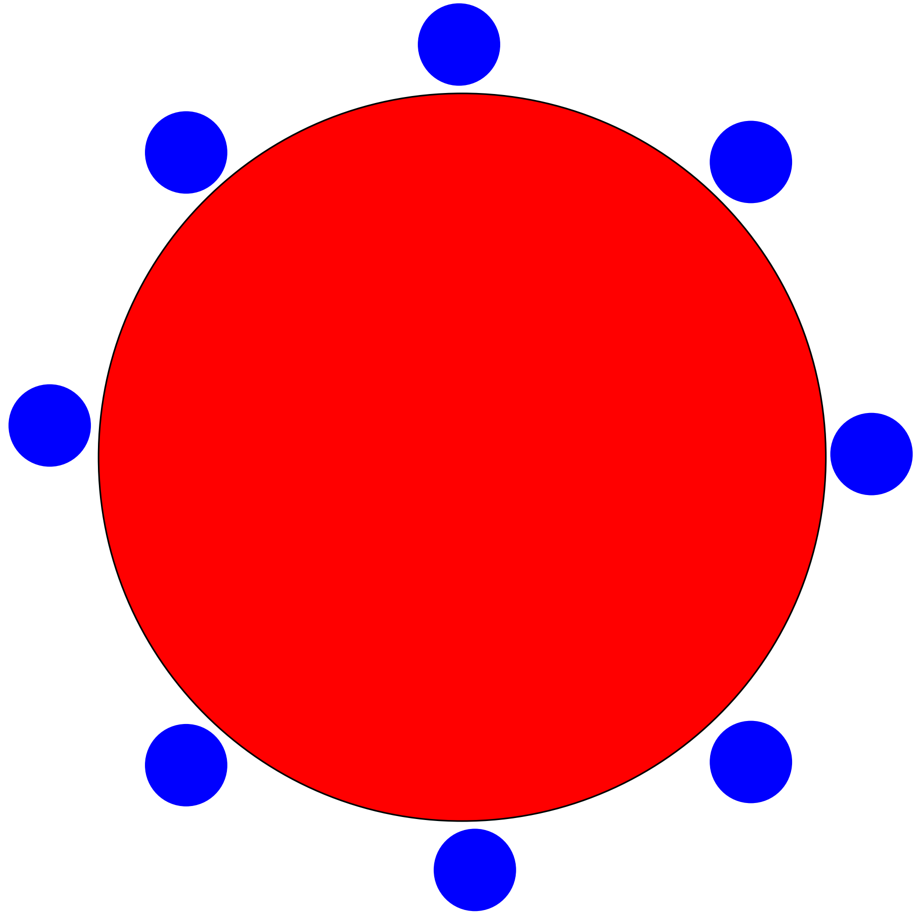
Harborth showed that from 10 points we can always choose an empty pentagon. Horton constructed sets of arbitrary large cardinality which do not contain any empty heptagon. Erdős's question for $n = 6$ was answered positively by Gerken and Nicolas.

Let $\mathcal{F}$ be a family of disjoint compact convex sets. A member A of $\mathcal{F}$ is a *vertex* of $\mathcal{F}$ if it is not contained in the convex hull of the union of the sets belonging to $\mathcal{F} \setminus \{A\}$. A sub-family $\underline{\mathcal{F}} \subset \mathcal{F}$ is in *convex position* if all of its members are vertices of $\underline{\mathcal{F}}$.

Theorem (Bisztriczky, GFT). *For every* $n \geq 4$ *there exists a smallest natural number* $g(n)$ *such that any family of at least* $g(n)$ *disjoint compact convex sets with the property that every three members of it are in convex position contains* n *members in convex position.*

It is an open question whether $f(n) = g(n)$. The best upper bound for $g(n)$ was obtained by Hubard, Montejano, Mora and Suk:

$$g(n) \leq \left(\binom{2n-5}{n-2} + 1 \right) \binom{2n-4}{n-2} + 1.$$



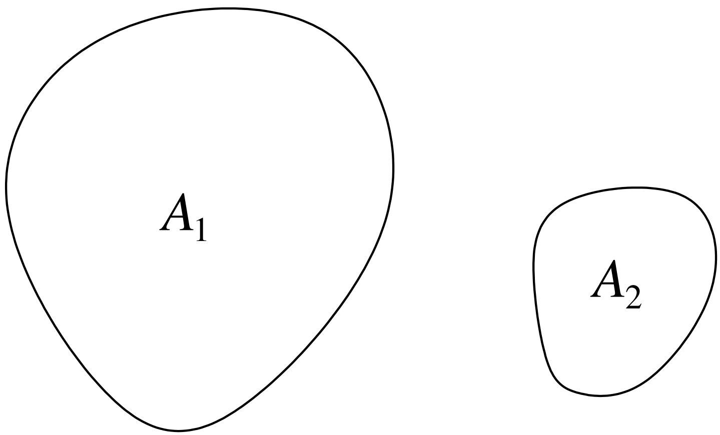
A family $\mathcal{F}$ *of convex discs satisfies property* P_k if any $m \leq k$ members of $\mathcal{F}$ are in convex position.

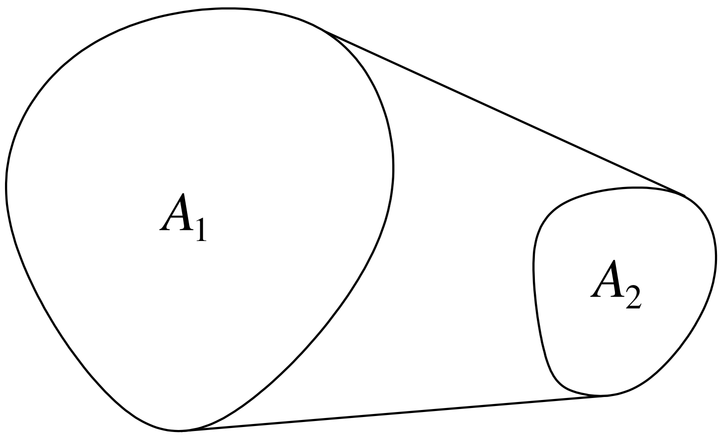
Theorem (Bisztriczky, GFT). *For every pair of integers* $k \geq 3$ *and* $n \geq 4$ *there exists a smallest natural number* $h(k, n)$ *such that any family of at least* $h(k, n)$ *disjoint compact convex sets satisfying property* P_k *contains* n *members in convex position.*

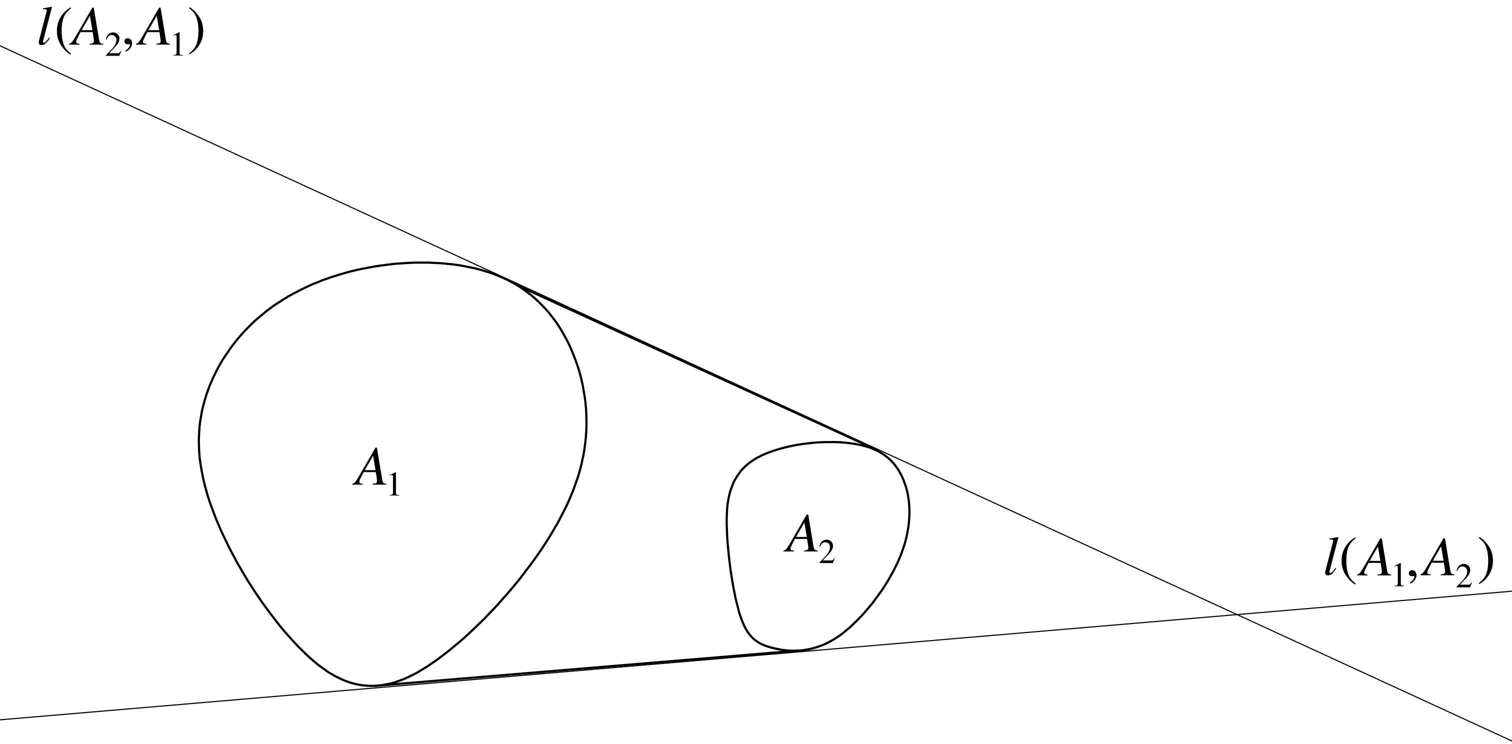
A sub-family $\underline{\mathcal{F}}$ of $\mathcal{F}$ is in *empty convex position in* $\mathcal{F}$ if it is in convex position and the convex hull of the union of its members does not contain any member of $\mathcal{F} \setminus \underline{\mathcal{F}}$. n members of $\mathcal{F}$ that are in empty convex position is called an *n-hole*.

Theorem. *For every pair of integers* $k \geq 4$ *and* $n \geq k$ *there is an integer* N *such that any family of more than* N *disjoint compact convex sets satisfying property* P_k *has* n *members in empty convex position.*

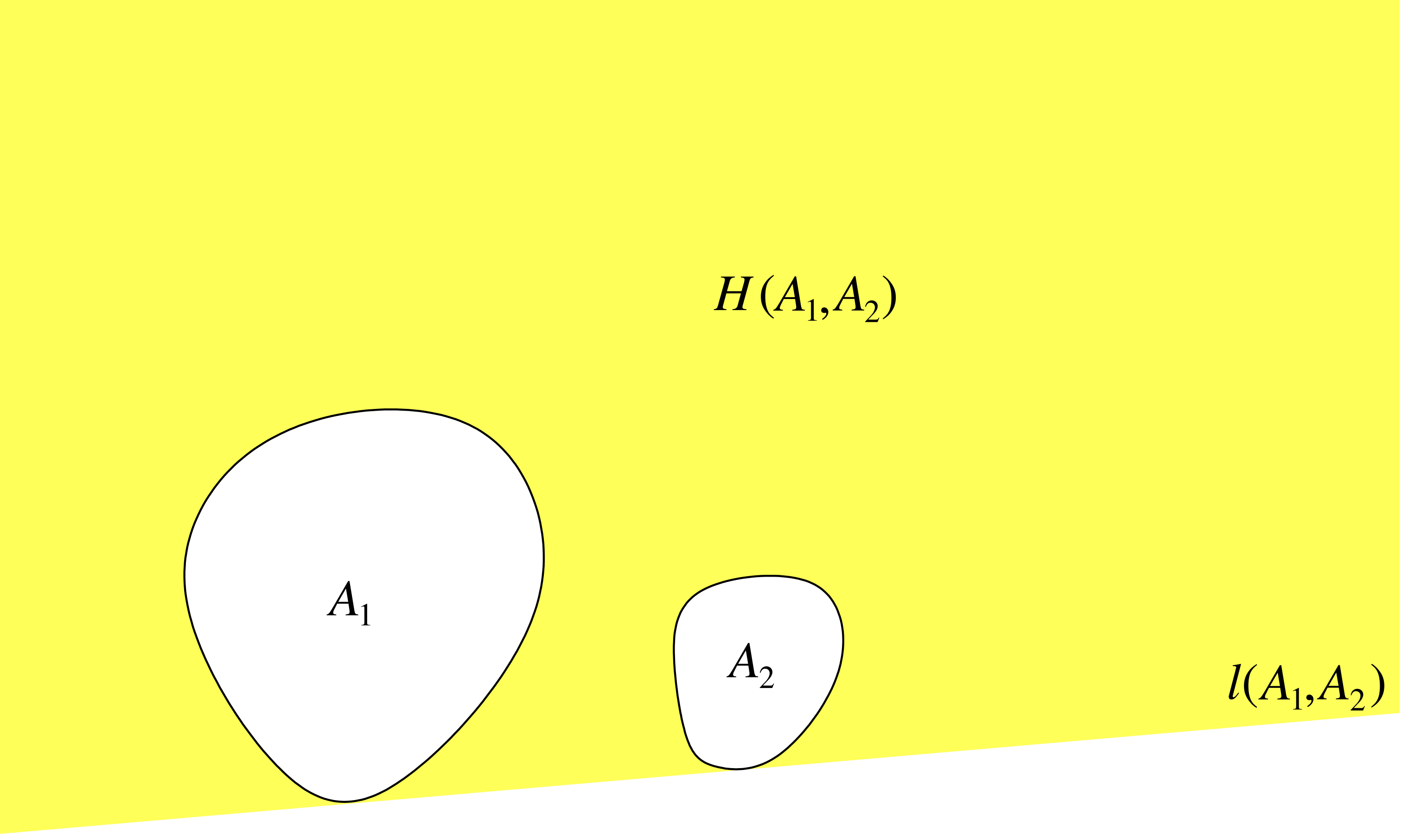
Let A_1 and A_2 be two disjoint compact convex sets with non-empty interiors. We observe that there are two distinct supporting lines $l(A_1, A_2)$ and $l(A_2, A_1)$ of $\operatorname{conv}(A_1 \cup A_2)$ which also support A_1 and A_2. We choose the notation so that while traveling on $l(A_i, A_j)$ so that $\operatorname{conv}(A_1 \cup A_2)$ is to the left, we meet first A_i and then A_j.


A_1
A_2

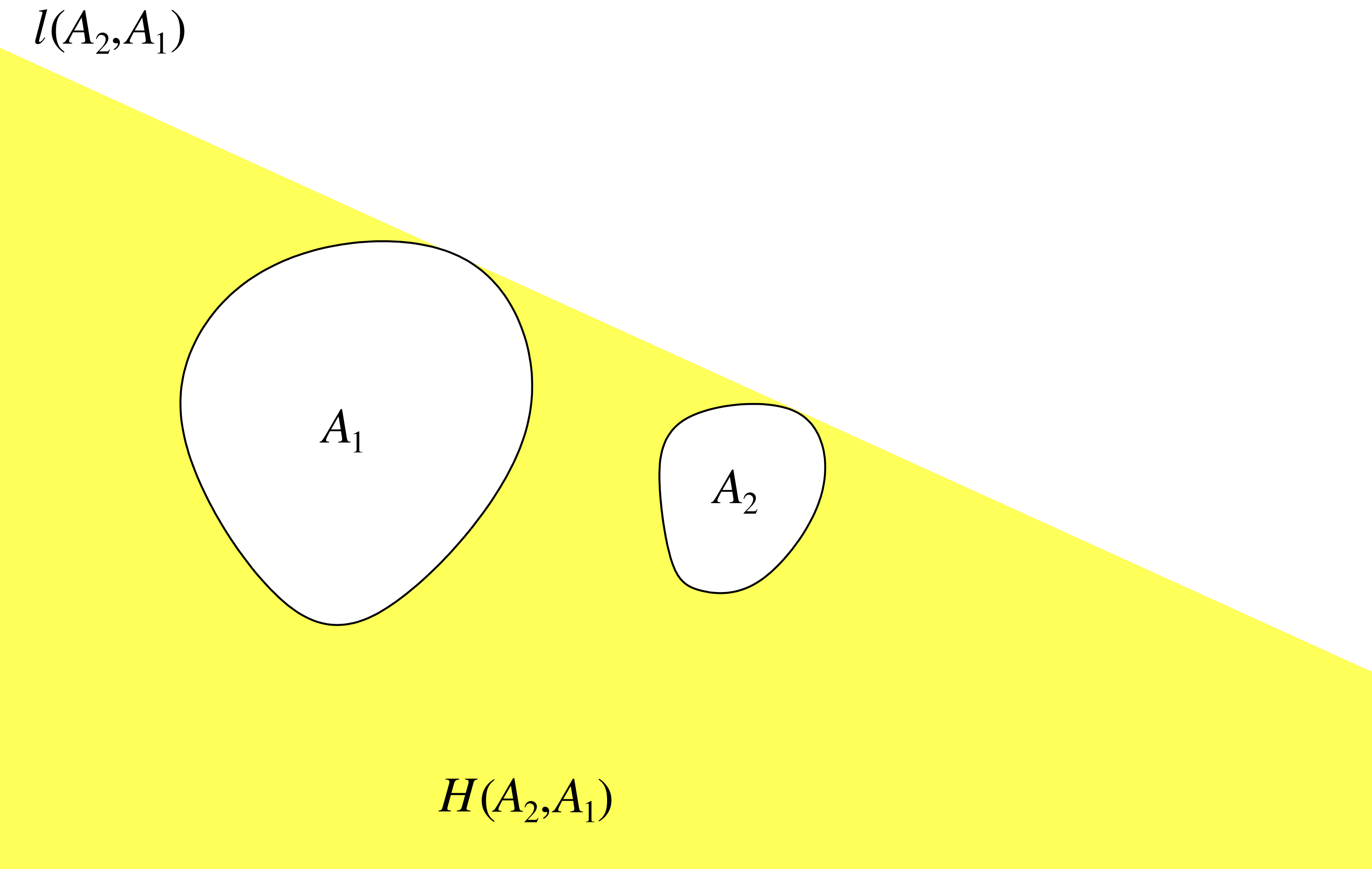

A_1
A_2

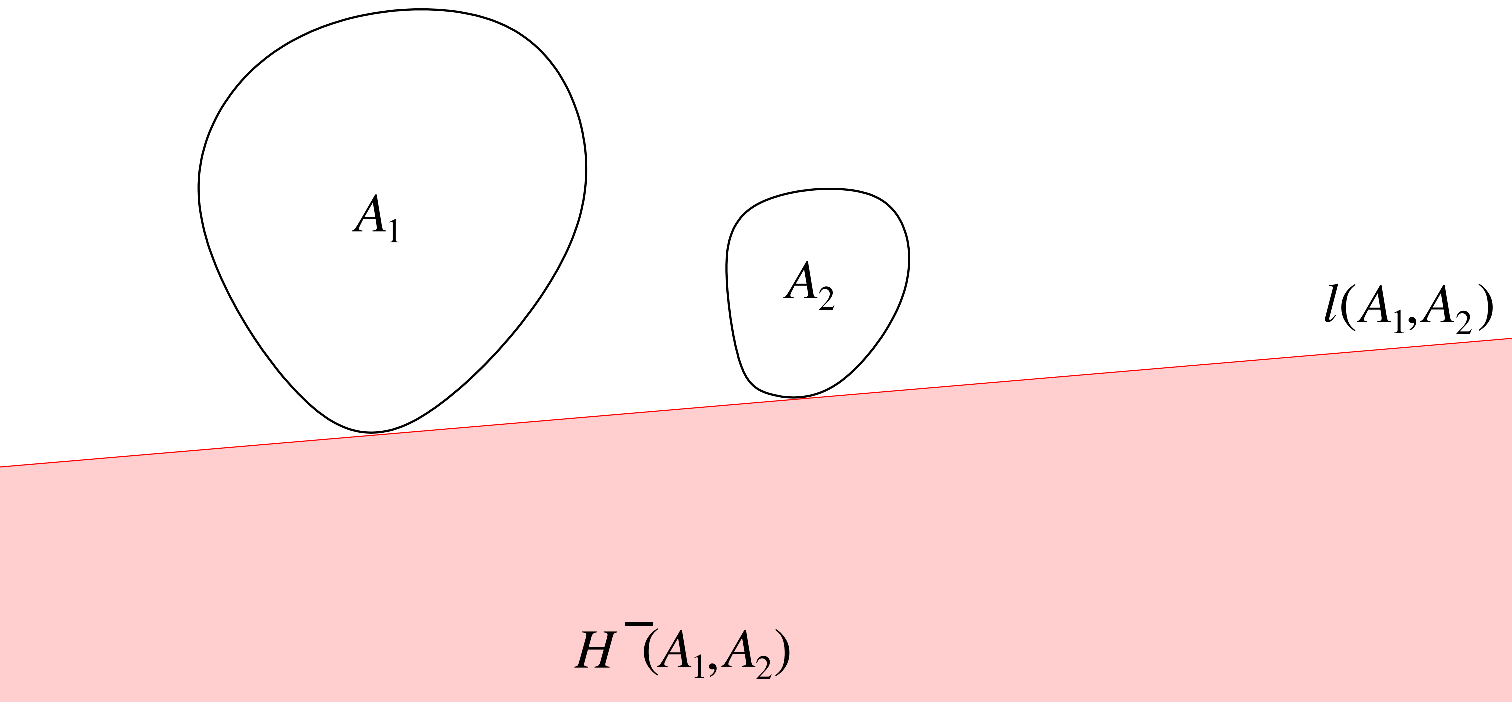

$l(A_2,A_1)$
A_1
A_2
$l(A_1,A_2)$

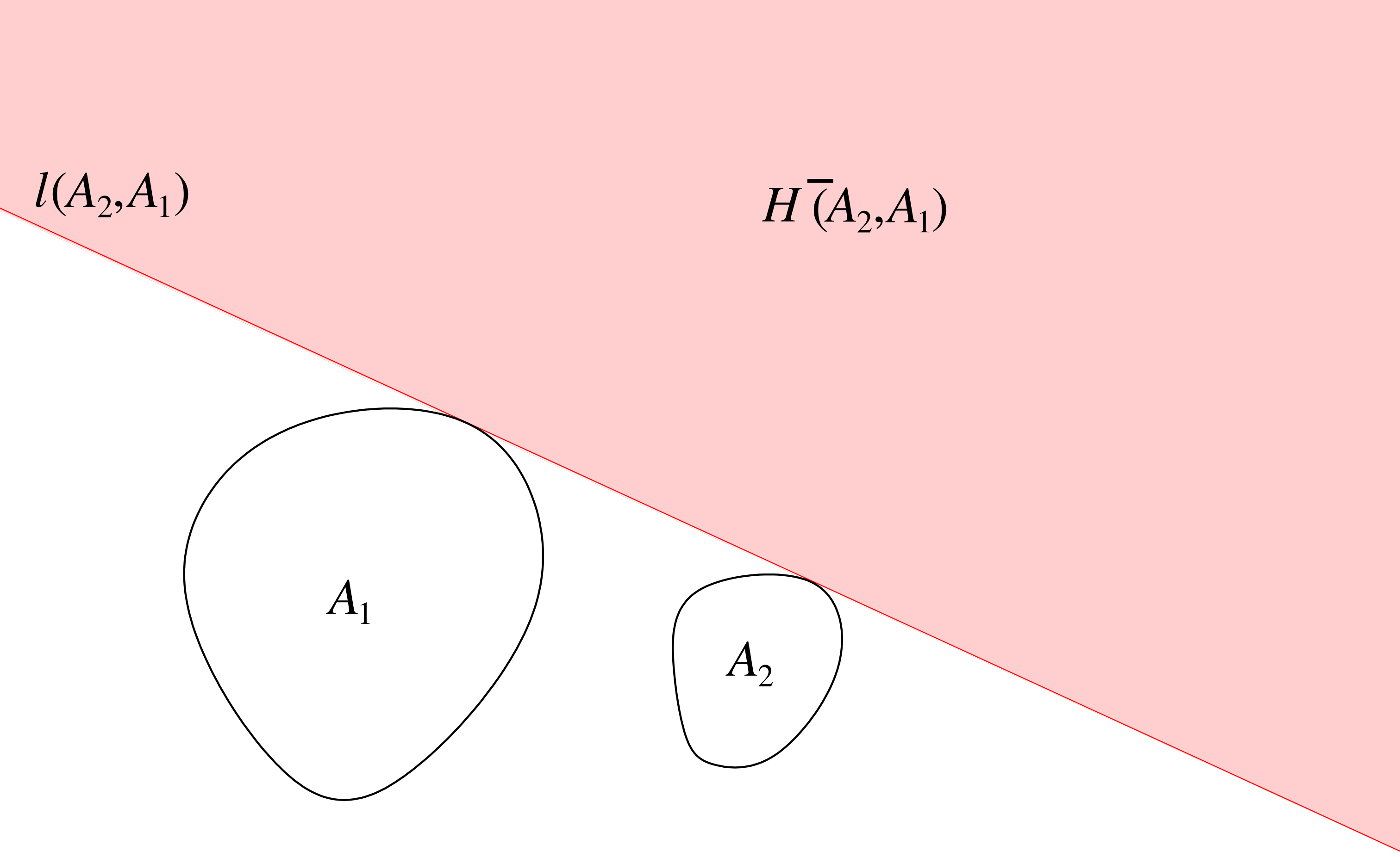
Let $H(A_i, A_j)$ be the open half-plane bounded by $l(A_i, A_j)$ containing $\operatorname{int}\operatorname{conv}(A_1 \cup B_2)$ and let $H^-(A_i, A_j)$ be the complement of $H(A_i, A_j)$.



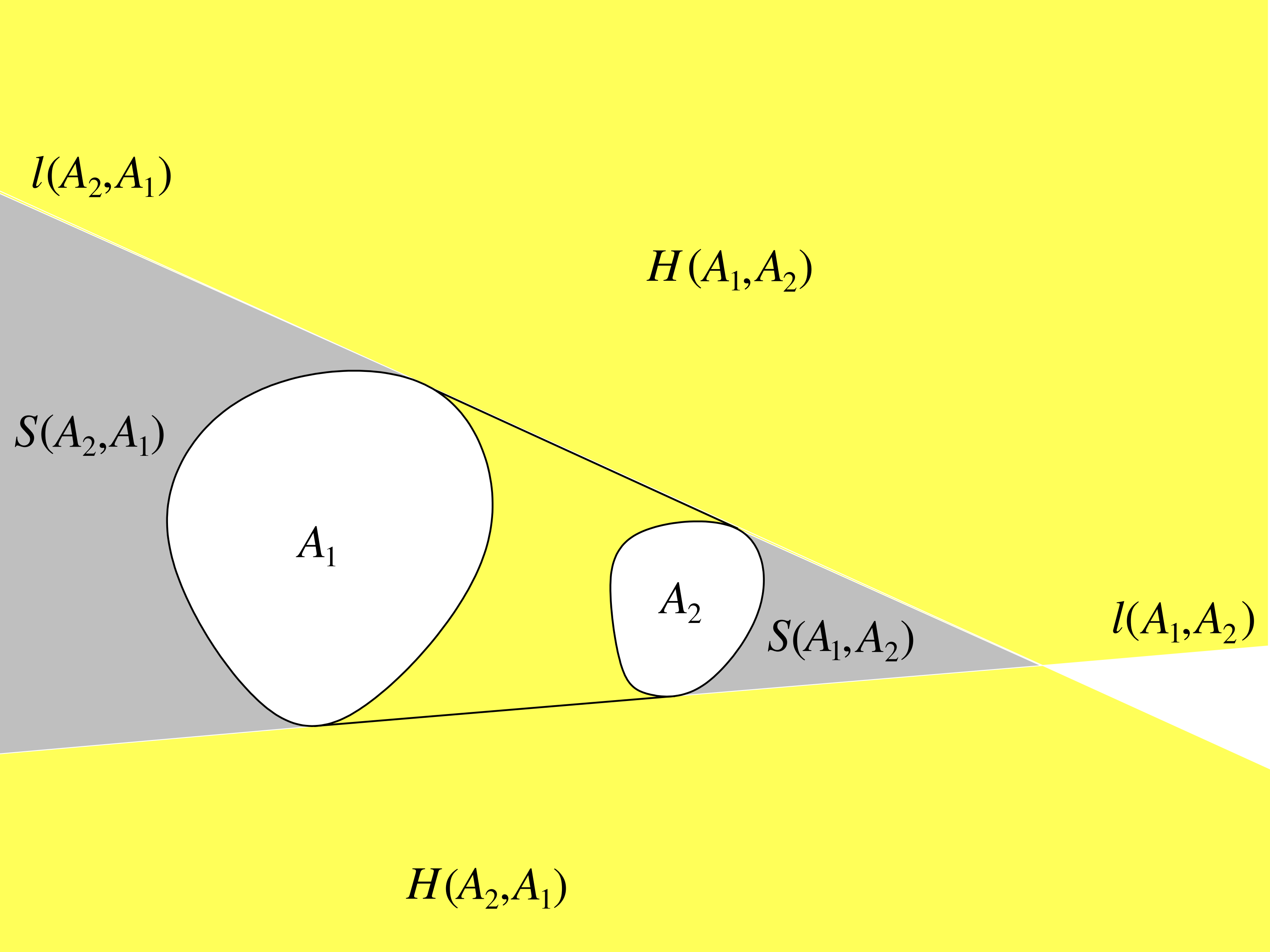
$H(A_1, A_2)$
A_1
A_2
$l(A_1, A_2)$


$l(A_2, A_1)$
A_1
A_2
$H(A_2, A_1)$


A_1
A_2
$l(A_1, A_2)$
$H^-(A_1, A_2)$

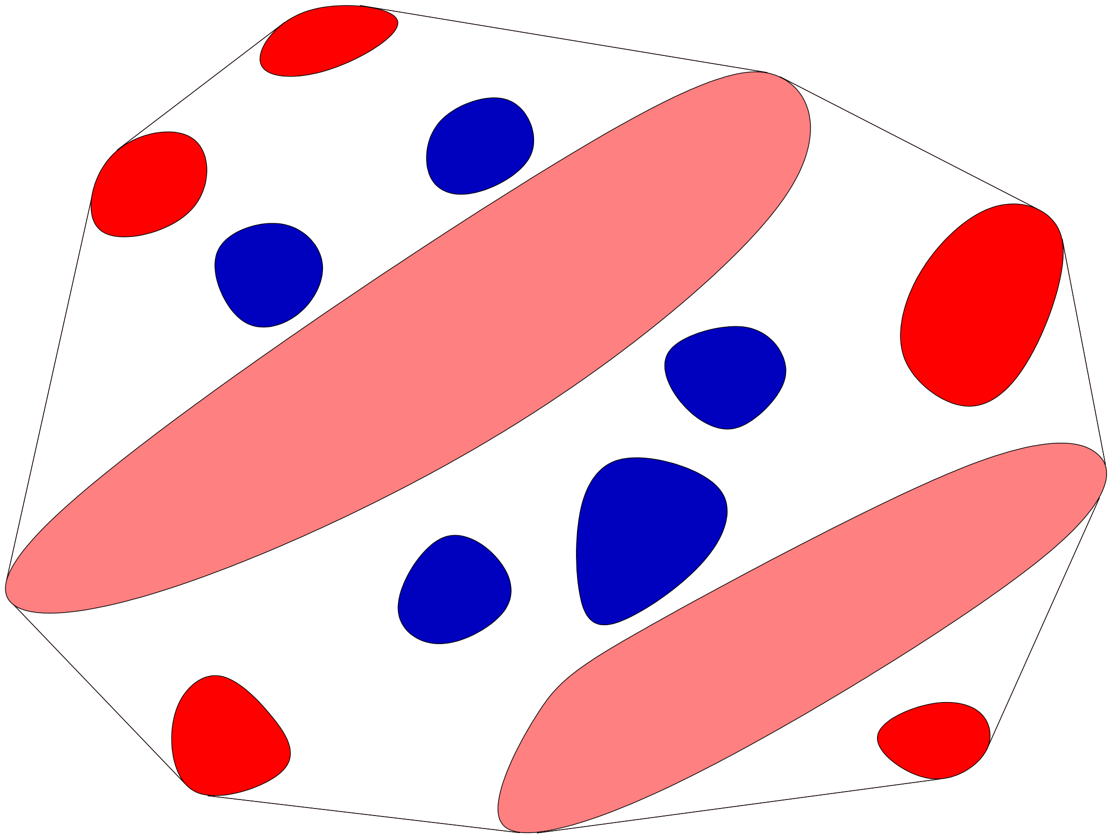

l(A2,A1)
H‾(A2,A1)
A1
A2

Let $L(A_1A_2) = L(A_2A_1) = H(A_1A_2) \cap H(A_2A_1)$ and define $S(A_i, A_j)$ as that component of $L(A_1A_2) \setminus \operatorname{int}\operatorname{conv}(A_1 \cap A_2)$ which has nonempty intersection with A_j. We call $S(A_i, A_j)$ the *shadow of* A_j *from* A_i.

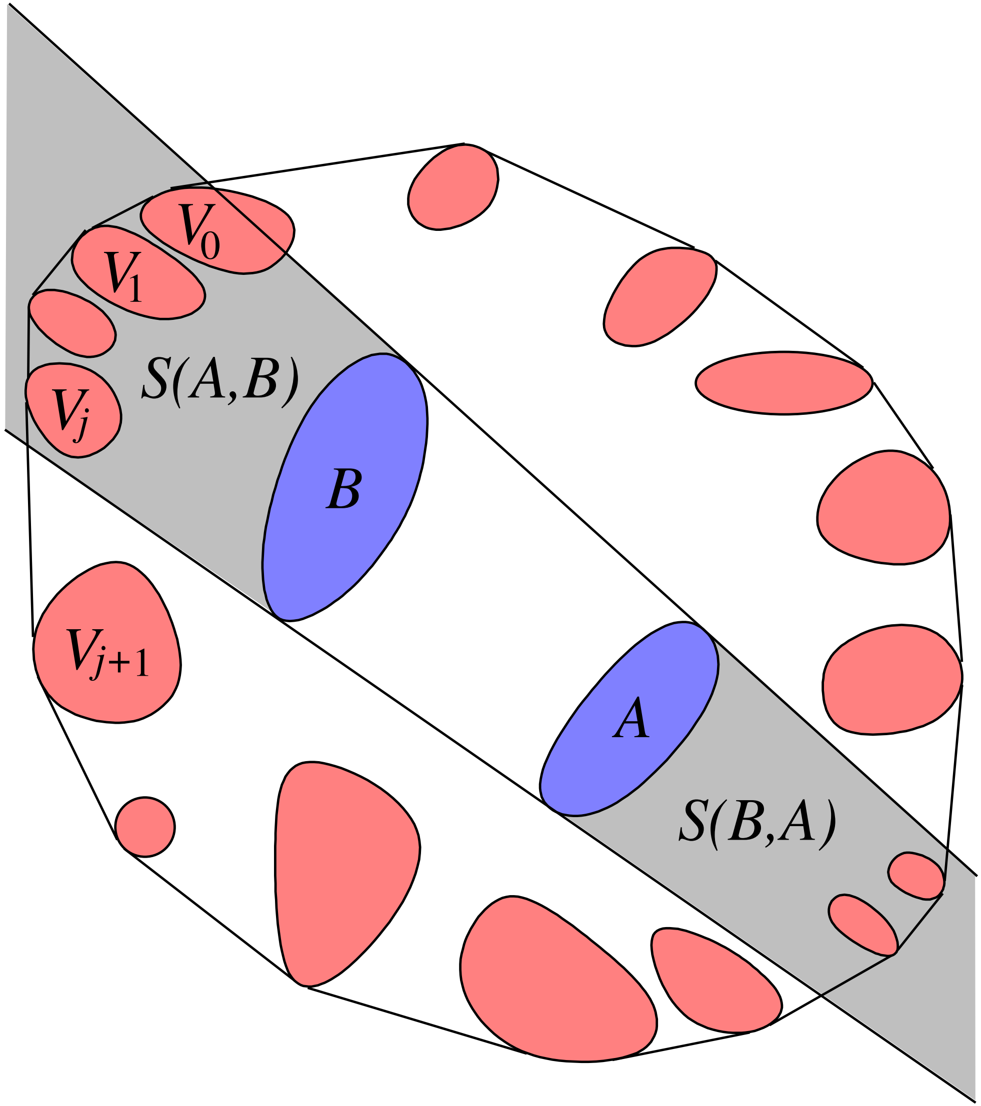


$l(A_2,A_1)$
$H(A_1,A_2)$
$S(A_2,A_1)$
A_1
A_2
$S(A_1,A_2)$
$l(A_1,A_2)$
$H(A_2,A_1)$

Members of $\mathcal{F}$ that are not vertices of $\mathcal{F}$ are called *internal members* of $\mathcal{F}$. A vertex V is called *regular* if $\operatorname{bd} V \cap \operatorname{conv} \bigcup_{A \in \mathcal{F}} A$ is connected, otherwise it is said to be *irregular*. The sub-families of $\mathcal{F}$ consisting of the vertices and of the internal embers of $\mathcal{F}$ are denoted by $\mathcal{V}(\mathcal{F})$ and $\mathcal{I}(\mathcal{F})$, respectively.

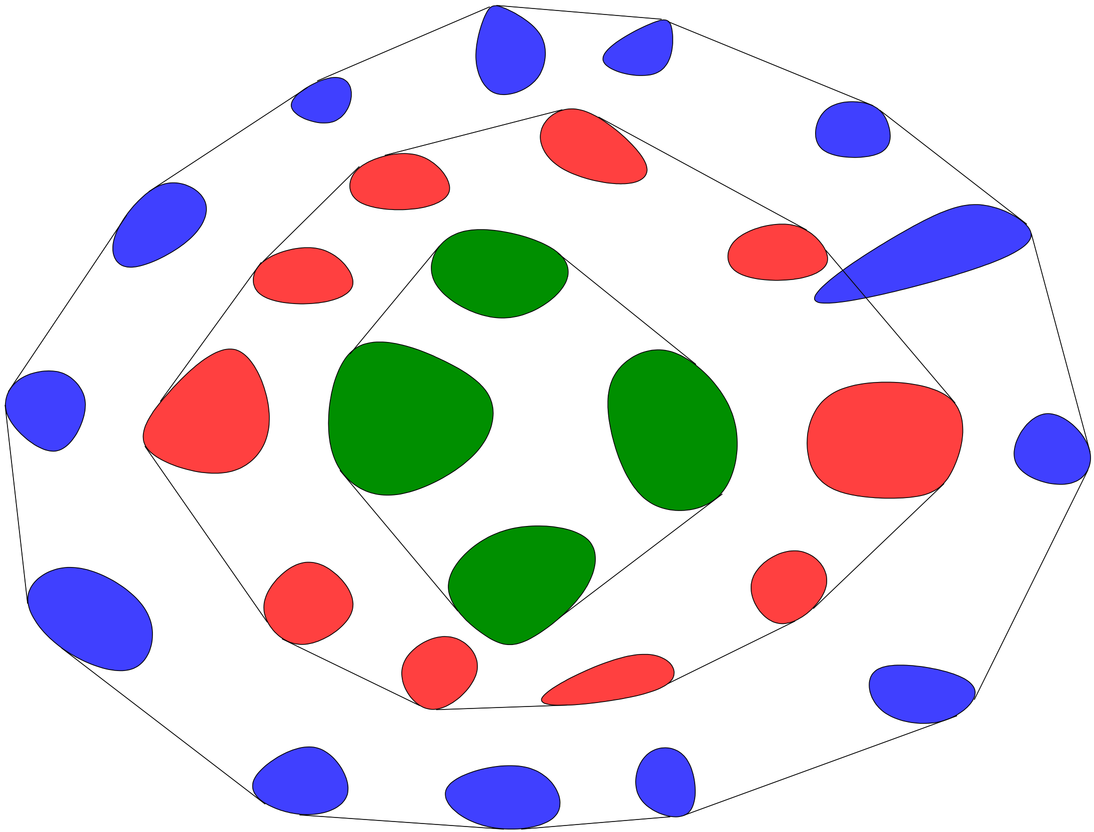


Lemma 1 (Géza Tóth). *Let* $k \geq 4$ *and let* $\mathcal{F}$ *be a family of disjoint compact convex sets satisfying property* P_k*. Let* A *be an arbitrary and* B *an internal member of* $\mathcal{F}$*. Then* $S(A, B)$*, contains at least* $k-3$ *vertices of* $\mathcal{F}$*.*

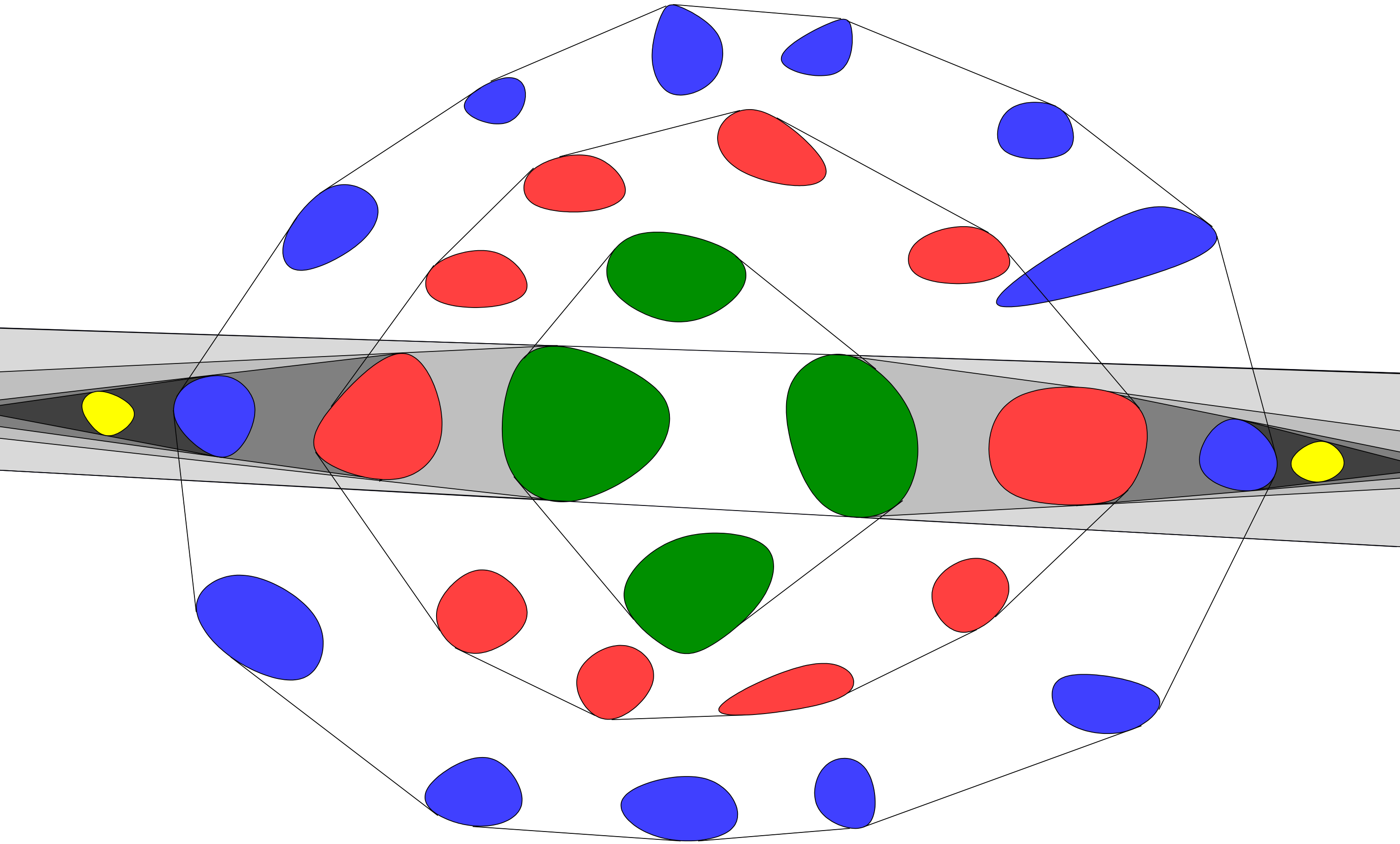


V_0
V_1
V_j
$S(A,B)$
B
V_{j+1}
A
$S(B,A)$

Write $\mathcal{F}_1 = \mathcal{F}$ and define inductively the families $\mathcal{F}_i$ and $\mathcal{S}_i$, $i = 1, \ldots, N$, by $\mathcal{F}_i = \mathcal{I}(\mathcal{F}_{i-1})$, $\mathcal{S}_i = \mathcal{V}(\mathcal{F}_i)$, $\mathcal{F}_N = \mathcal{V}(\mathcal{F}_N) = \mathcal{S}_N$. We shall refer to the families $\mathcal{F}_i$ and $\mathcal{S}_i$ as the *i-th core* and the *i-th shell of* $\mathcal{F}$, *respectively.*



Lemma 2. *Let $k \geq 4$ and let $\mathcal{F}$ be a family of disjoint compact convex sets satisfying property P_k. Suppose that $\mathcal{F}$ consists of N shells. If $\mathcal{S}_N$ contains more than one members, then $\mathcal{F}$ contains a $2N$-hole. If $\mathcal{S}_N$ consists of a single member, then $\mathcal{F}$ contains $(2N-1)$ members in empty convex position*

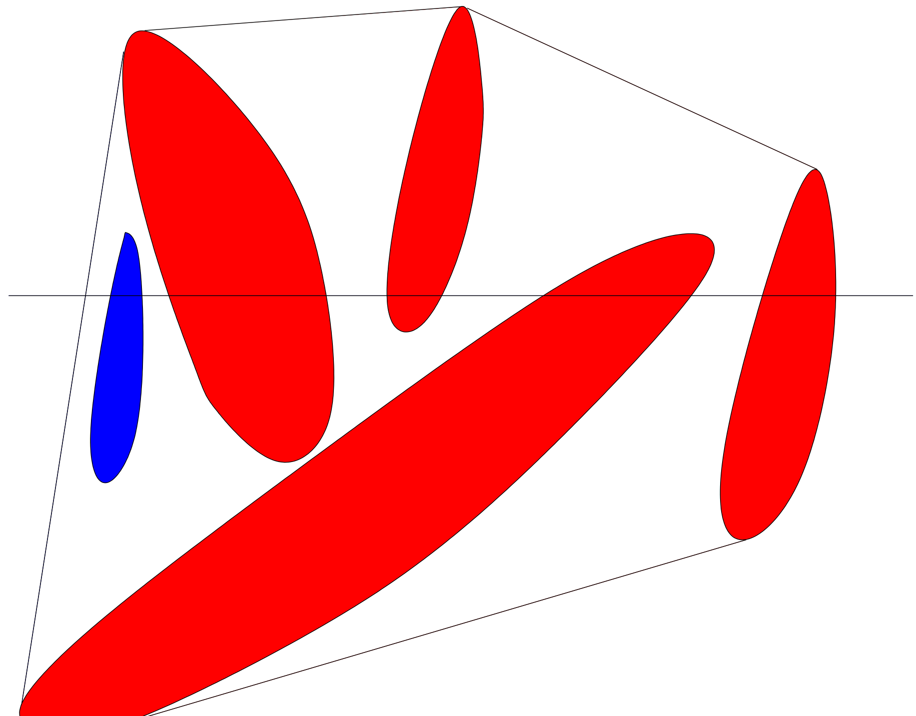


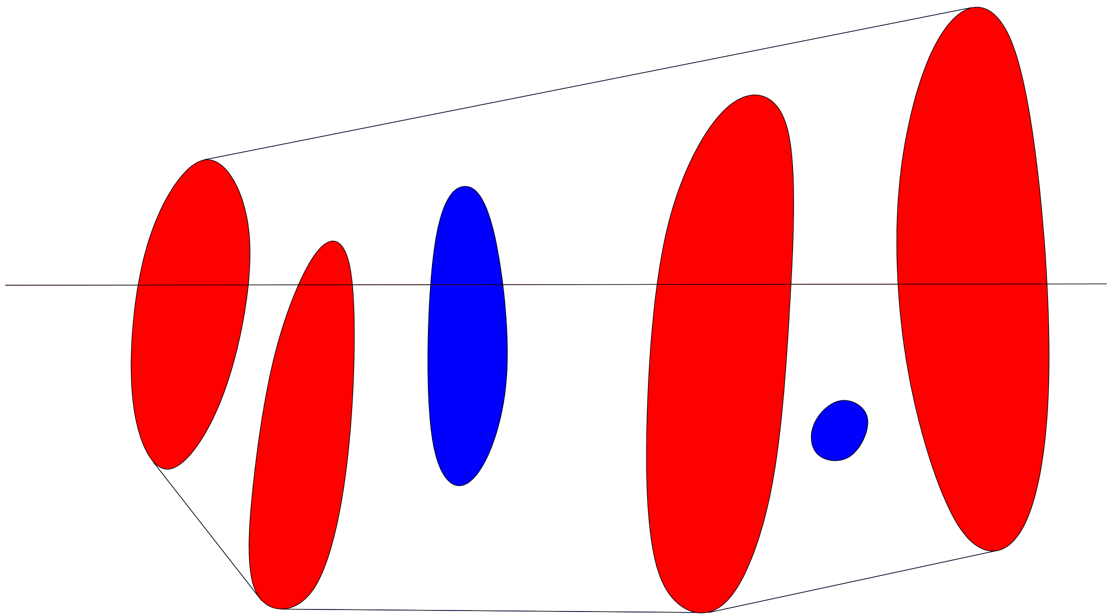
Lemma 3. *Let* $\mathcal{H}$ *be a family of disjoint compact convex sets and let* $\underline{\mathcal{H}}$ *be the sub-family of* $\mathcal{H}$ *consisting of all members of* $\mathcal{H}$ *which intersect a straight segment* L.

(i) *If* $\mathcal{H}$ *satisfies property* P_5, *then* $\underline{\mathcal{H}}$ *is in empty convex position in* $\mathcal{H}$.

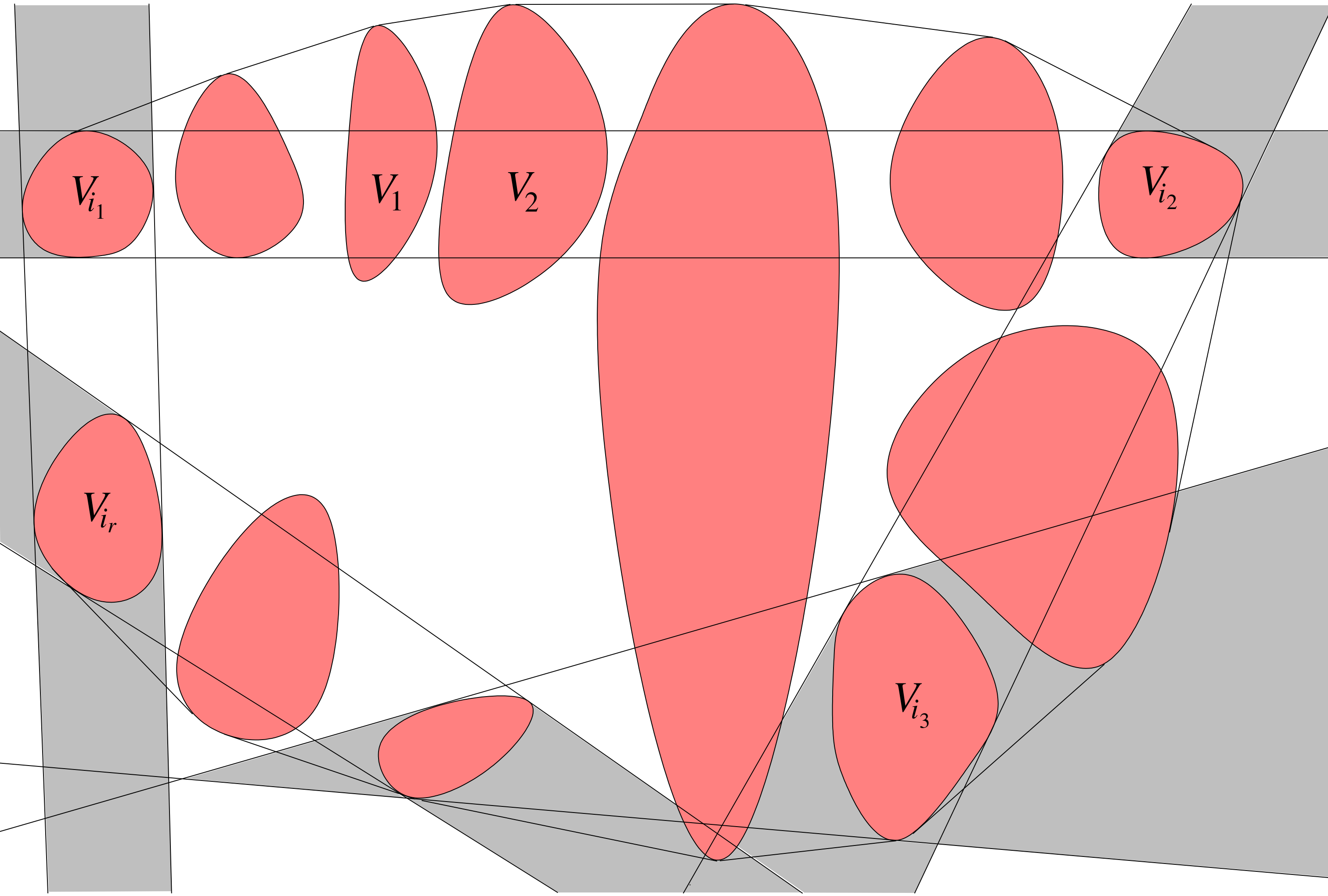
(ii) *If* $\mathcal{H}$ *satisfies property* P_4 *and*

Proposition. *Suppose that all members of a family* $\mathcal{F}$ *of disjoint compact convex sets intersect a line* L*. Then the first, as well as the last member of* $\mathcal{F}$ *intersecting* L *is a vertex of* $\mathcal{F}$*. If a convex set* B *which is disjoint from all members of the family is contained in* $\operatorname{conv} \bigcup_{A \in \mathcal{F}} A$*, then* B *is contained in the convex hull of the union of at most four members of* $\mathcal{F}$*. Moreover, if* B *lies in one of the closed half-planes bounded by* L*, then it is contained in the convex hull of the union of two members of* $\mathcal{F}$*.*





Lemma 4. *Let $\mathcal{F}$ be a family of disjoint compact convex sets satisfying property P_k for some $k \geq 3$. Let $V_1, \ldots, V_m$ be the regular vertices of $\mathcal{F}$ enumerated in the cyclic order as we meet them while traveling on* $\operatorname{bd}\operatorname{conv}\bigcup_{A\in\mathcal{F}} A$ *in clockwise direction. Then one can choose $l \leq m$ regular vertices $V_{i_1}, \ldots, V_{i_l}$ of $\mathcal{F}$ enumerated in their natural cyclic order such that $V_l \cap H^-(V_{i_j}, V_{i_{j+1}}) \neq \emptyset$ for $l = i_j, i_j + 1, \ldots, i_{j+1} - 1, i_{j+1}$ and no member of $\mathcal{F}$ is contained in $S(V_{i_j}, V_{i_{j+1}})$ or $S(V_{i_j}, V_{i_{j-1}})$, $j = 1, \ldots, l$, $V_{i_{j\pm l}} = V_{i_j}$.*



V_{i_1}
V_1
V_2
V_{i_2}
V_{i_r}
V_{i_3}

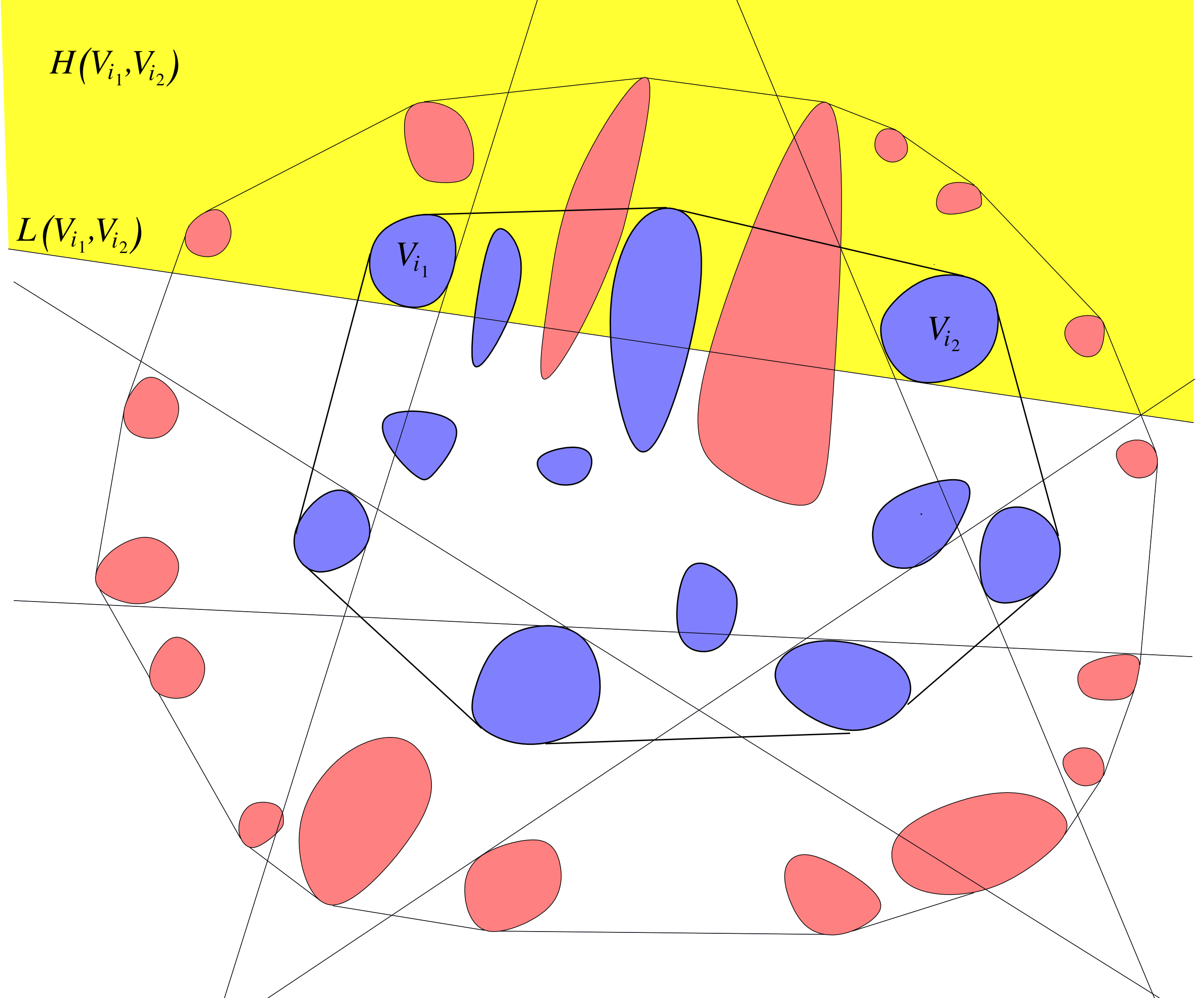
Lemma 5. *Let $k \geq 3$ be an integer and let $\mathcal{F}$ be a family of disjoint compact convex sets satisfying property P_k for some $k \geq 4$. Let $\overline{\mathcal{F}}$ be a sub-family of $\mathcal{F}$ with M vertices and let $\underline{\mathcal{F}}$ be the sub-family of $\overline{\mathcal{F}}$ consisting of its internal members. If $\underline{\mathcal{F}}$ has $m \geq 2$ regular vertices,*

(i) $k \geq 5$, *and* $M \geq 2m(2n - k)$,

or if

(ii) $k = 4$ *and* $M \geq m[(n-3)|\underline{\mathcal{F}}| + 4]$.

then $\mathcal{F}$ has n members in empty convex position.


$H(V_{i_1},V_{i_2})$
$L(V_{i_1},V_{i_2})$
V_{i_1}
V_{i_2}

Proof of the Theorem. Suppose that the family $\mathcal{F}$ of disjoint compact convex sets satisfies property P_k for $k \geq 4$ and does not contain any n-hole. Let

$$a_i = |\mathcal{F}_i|$$

be the number of members in the i-th core $\mathcal{F}_i$. Then

$$|\mathcal{S}_i| = a_i - a_{i+1} \quad \text{for} \quad i = 1, \ldots, N-1.$$

Since $\mathcal{F}$ does not contain any n-hole, we have

$$a_N \leq n - 1$$

It follows from Lemma 2 that $\mathcal{F}$ consists of at most $\lfloor \frac{n+1}{2} \rfloor$ shells, thus

$$N \leq \lfloor \frac{n+1}{2} \rfloor.$$

We apply Lemma 5 to $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ in the roles of $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$.

$$a_i - a_{i+1} \le [(n-3)a_i + 4](a_{i-1} - a_i) - 1, \quad 1 \le i \le N-1$$

if $k = 4$ and

$$a_i - a_{i+1} \le (2n-k)(a_{i-1} - a_i) - 1, \quad 1 \le i \le N-1$$

if $k > 4$.

We do not expect that the exact solution of these recursions gives a sharp bound for $e(k,n)$. Therefore we use the following weaker inequalities that are easily obtained from the above ones.

$$a_i \leq n a_{i+1}^2 \quad \text{for} \quad 1 \leq i \leq N-1 \tag{1}$$

if $k = 4$ and

$$a_i \leq (2n-k+1) a_{i+1} \quad \text{for} \quad 1 \leq i \leq N-1 \tag{2}$$

if $k > 4$.

Hence it follows that

$$|\mathcal{F}_i| = |\mathcal{F}_i| = a_1 \leq a_N \leq a_{\frac{n+1}{2}} \leq n^{2^n}$$

if $k = 4$ and

$$|\mathcal{F}_i| = |\mathcal{F}_i| = a_1 \leq a_N \leq (n-1)(2n-k+1)^{\frac{n+1}{2}}$$

if $k \geq 5$.