6/28/11

The Approximability of Constraint Satisfaction Problems
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Defn| (α, β)-approximability $CSP(\Gamma)$: if $Opt(\mathcal{I}) \geq \beta$ alg's assignment has value $\geq \alpha$.

Basic LP| max $LPVal_{\mathcal{I}}(\mathcal{L}) := \underset{C=(R,S)\in\mathcal{I}}{avg} \{ \Pr_{L\sim\lambda_C}[L(S) \in R] \}$

s.t. λ_C is prob. dist. on local assignments $S \to D$ $\forall$ $C = (R, S)$, μ_v is prob. dist. on D $\forall v \in V$,

(‡) $\Pr_{L\sim\lambda_C}[L(v) = \ell] = \mu_v[\ell]$ $\forall$ C $\forall$ $v \in S$ $\forall$ $\ell \in D$.

Thm| Given $\mathcal{I}$ for max-(k)Sat

- Solve LP optimally, get $\mathcal{L} = ((\mu_v), (\lambda_C))$
- Output $F : V \to \{0, 1\}$: $F(v) \sim \mu_v$ independently $\forall$ $v \in V$

Then this $((1 - 1/e)\beta, \beta)$-approxes $\forall$ β.

eg| $C = x \vee \bar{y} \vee z$. $\Pr_F[F \text{ satisfies } C] = 1 - \mu_x[0] \cdot \mu_y[1] \cdot \mu_z[0]$

Generally say $C = (R, S)$
$b_C : S \to \{0, 1\}$ for the bad (forbidden) assign. for C.

$$\Pr_F[F \text{ satisfies } C] = 1 - \prod_{v \in S} \mu_v[b_C(v)] \quad (*)$$

$$p_C := \Pr_{L\sim\lambda_C}[L(S) \text{ satisfies } R] \leq \sum_{v\in S} \Pr_{L\sim\lambda_C}[L(v) \neq b_C(v)]$$

$$\Pr_{L\sim\lambda_C}[L(v) \neq b_C(v)] = \mu_v[\overline{b_C(v)}]$$

$$= \sum_{v\in S} (1 - \mu_v[b_C(v)]) \quad (\text{by } (‡)).$$

$$= |S| \underset{v\in S}{AM} \{1 - \mu_v[b_C(v)]\}$$

$$(*) = 1 - \underset{v\in S}{GM} \{\mu_v[b_C(v)]\}^{|S|}$$

$$\geq 1 - AM\{\mu_v[b_C(v)]\}^{|S|}$$

$$= 1 - (1 - \underset{v\in S}{AM}\{1 - \mu_v[b_C(v)]\})^{|S|}$$

$$\geq 1 - \left(1 - p_C/|S|\right)^{|S|} \geq \left(1 - \frac{1}{e}\right) p_C$$

notes by Megan Heenehan

$\therefore$ $\Pr[F \text{ satisfies } C] \geq \left(1 - \frac{1}{e}\right) p_c$

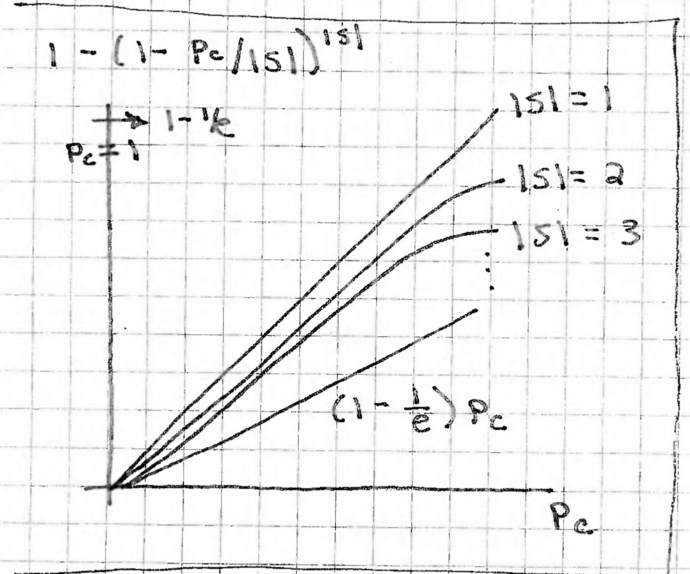
$\text{LPVal}(\mathcal{L}) = \text{avg}\{p_c\}$

$\therefore$ $\mathop{E}_{F}[\text{Val}(F)] \geq \text{avg}\left\{\left(1 - \frac{1}{e}\right) p_c\right\}$

$\geq \left(1 - \frac{1}{e}\right) \text{LPVal}(\mathcal{L})$

$\square$

<u>Cor</u>] For Max-2 Sat, $(3/4\,\beta, \beta)$-approx.



<u>Cor</u>] For Max-≥2 Sat $\left(\frac{3}{4}\beta, \beta\right)$-approx. with trivial random alg.

$\left(\frac{7}{8} + \varepsilon, 1\right)$-hard for E3-Sat.

<u>Cor</u>] For $\mathcal{I}$, instance of Max-Sat
$\text{Opt}(\mathcal{I}) \leq \text{LP Opt}(\mathcal{I}) \leq \frac{4}{3}\text{Opt}(\mathcal{I})$.

<u>Defn</u>] <u>(α, β)-gap instance</u> for $\text{CSP}(\Gamma)$ is $\mathcal{I}$ s.t.
$\text{Opt}(\mathcal{I}) \leq \alpha$, but $\text{LPOpt}(\mathcal{I}) \geq \beta$

(*) If such $\mathcal{I}$ exists basic LP is <u>not</u> (α, β)-dist. alg.

Prop] This is a $(3/4, 1)$-gap instance for E2-Sat
- $\underset{c_1}{x \vee y}$, $\underset{c_2}{\bar{x} \vee y}$, $\underset{c_3}{x \vee \bar{y}}$, $\underset{c_4}{\bar{x} \vee \bar{y}}$

Check: $\text{Opt} \le 3/4$ ✓

LP Sol: μ_x: $\frac{1}{2}/\frac{1}{2}$ on $0, 1$ μ_y same

λ_{c_2}: $\frac{1}{2}$ prob. on $(0,0)$, $\frac{1}{2}$ prob. on $(1,1)$

Similarly for c_1, c_3, c_4
Get $\text{LPOpt} = 1$.

Max-Cut $D = \{0,1\}$ $\Gamma = \{\neq\}$
$\forall G$ (G a graph) $\text{LPOpt}(G) = 1$

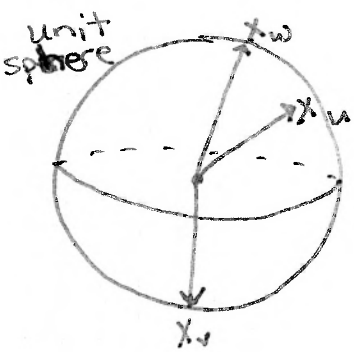
$\forall v \in V$ μ_v: $\frac{1}{2}$ on 0, $\frac{1}{2}$ on 1

$\forall e \in E$ λ_e: $\frac{1}{2}$ on $(0,1)$, $\frac{1}{2}$ on $(1,0)$

(GW)
Semidefinite Programming (SDP) Relaxation for instance $G = (V, E)$ of Max-Cut

Maximize $\text{SDPVal}(\mathcal{X}) = \underset{(u,v)\in E}{\text{avg}} \left\{ \frac{1}{2} - \frac{1}{2}\langle X_u, X_v \rangle \right\}$

s.t. $\mathcal{X} = (X_v)_{v \in V}$ a collection of n-dim. vectors with $\langle X_v, X_v \rangle = \|X_v\|_2^2 = 1$ (n = # vertices)
$X_v \in \mathbb{R}^n$



Prop] $\text{Opt}(G) \le \text{SDPOpt}(G) \le \text{LPOpt}(G)$

Pf] Let $F: V \to \{-1, 1\}$ be optimal cut.
$\mathcal{X}$: Define $X_v = (F(v), \underbrace{0, 0, \dots, 0}_{n-1})$

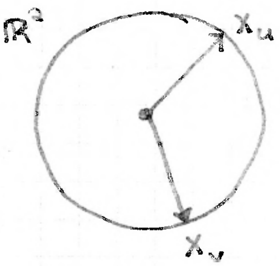
$\text{SDPVal}(\mathcal{X}) = \underset{(u,v)\in E}{\text{avg}} \left(\frac{1}{2} - \frac{1}{2} F(u)F(v) \right)$

$= \underset{(u,v)\in E}{\text{avg}} \begin{cases} 0 & \text{if } F(u) = F(v) \\ 1 & \text{if } F(u) \ne F(v) \end{cases}$

$= \text{Val}_G(F)$

<u>GW algorithm</u>: Pick a uniformly random hyperplane through 0 (the origin). (Pick its normal Z from a rotationally symmetric distribution.) Define $F(v) = \text{sign}(\langle Z, X_v \rangle)$.

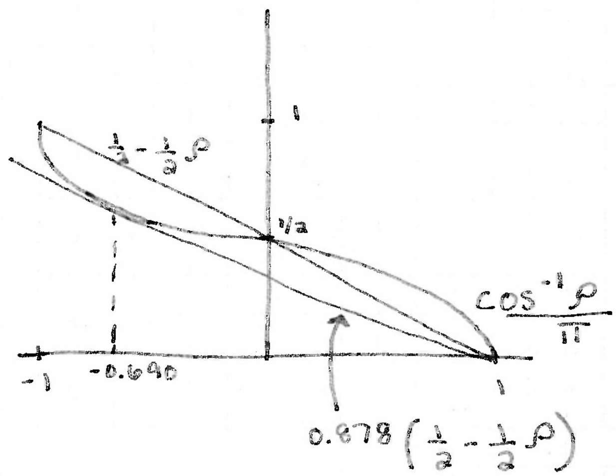
Fix $e = (u,v)$? $\Pr_Z[\text{alg "cuts" } e]$



$$= \frac{\text{angle}(X_u, X_v)}{\pi}$$

$$= \frac{\cos^{-1}(\langle X_u, X_v\rangle)}{\pi} \geq 0.878\left(\frac{1}{2} - \frac{1}{2}\langle X_u, X_v\rangle\right)$$

$\rho_{uv} := \langle X_u, X_v \rangle$



$$\mathop{E}_{GW}[\text{Val}_G(F)] = \mathop{\text{avg}}_{(u,v)\in E}\{\Pr[(u,v) \text{ cut}]\}$$

$$\geq \mathop{\text{avg}}_{(u,v)}\left\{0.878\left(\frac{1}{2} - \frac{1}{2}\langle X_u, X_v\rangle\right)\right\}$$

$$= 0.878 \text{ SDPVal}(\mathcal{X})$$

<u>Thm</u> $E[\text{Val}] \geq 0.878 \text{ SDPOpt}(G) \geq 0.878 \text{ Opt}(G)$

$\therefore$ $(0.878\beta, \beta)$ − approx $\forall \beta$

$(1 - O\sqrt{\varepsilon}, 1 - \varepsilon)$

<u>Thm</u> $\forall \beta \geq 0.845$ $\exists$ a graph G_n with $\text{SDPOpt}(G_n) \geq \beta - o_n(1)$ and $\text{Opt}(G_n) \leq 0.878\beta + o_n(1)$

Let G be graph whose vertices
$V = \{x \in \mathbb{R}^n \mid \|x\|_2^2 = 1\}$
$E = \{(x,y) \mid \langle x, y\rangle \leq \frac{1}{2} - \frac{1}{2}\beta\}$

See notes posted for check.

Univer

$\mathbb{K}_3 =$

$G_1 =$

G_1 i

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