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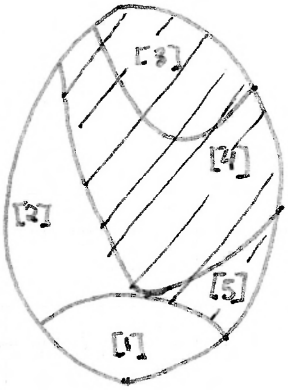
Universal Algebra for CSP Ross Willard



/// = "omit types 1, 2"

$\mathbb{G}$ fin. idemp. structure
$\underset{\sim}{A}$ = PolAlg($\mathbb{G}$)

HSP($\underset{\sim}{A}$) admit 1 or 2	HSP($\underset{\sim}{A}$) omit both 1, 2
(1) $\mathbb{G}_1 \le_{ppc} \mathbb{G}$ or $\exists$ fin. abelian group $(H, +)$ s.t. $(H; \text{"}x+y=z\text{"}, \{C_a : a \in H\}) \le_{ppc} \mathbb{G}$.	(1) $\underset{\sim}{A}$ satisfies some specific Maltsev condition (2) $\underset{\sim}{A}$ has a k-ary WNU for almost all k.

Bounded Width Conj (now Thm Barto, Kozik)
$\mathbb{G}$ idempotent. CSP($\mathbb{G}$) in P via k-consistency algorithm. $\Leftrightarrow$ HSP(PolAlg($\mathbb{G}$)) omits 1, 2.

Defn Let $(V, <)$ be a linearly ordered set.
$\vec{x} = (x_1, \dots x_k) \in V^k$ say $\vec{x}$ is proper if $x_1 < x_2 < \dots < x_k$.

Defn Given finite algebra $\underset{\sim}{A} = (A; \mathcal{F})$, $n \ge 1$,
$\mathbb{G}(\underset{\sim}{A}, n) = (A; \{\text{all } B \le \underset{\sim}{A}^k : 1 \le k \le n\})$ (B is the domain of a subalg. of $\underset{\sim}{A}^k$)
$CSP(\underset{\sim}{A}, n) := CSP(\mathbb{G}(\underset{\sim}{A}, n))$.

$CSP_0(\underset{\sim}{A}, n)$ means the subproblem of $CSP(\underset{\sim}{A}, n)$ (constraints version), which only considers instances $(V, \{\text{constraints}\})$ (V=variables)
- have a lin. order < on V
- Every constraint $((x_{i_1}, \dots, x_{i_k}), R)$ has proper scope
- $\forall\ 1 \le k \le n$, every proper k-tuple from $(V, <)$, $\vec{x}$ is the

notes by Megan Heenehan

scope of exactly one constraint

Idea) Instances of $CSP_0(\underset{\sim}{A}, n)$
V - set of variables

$x \in V$ $\quad \underset{\sim}{B}_x \leq \underset{\sim}{A}$ $\qquad\qquad x \in B_x$

$(x,y) \in V^2$ $\quad B_{x,y} \leq \underset{\sim}{A}^2$ $\qquad\qquad (x,y) \in B_{x,y}$
$x < y$

$\vdots$

$(x_1, \ldots, x_n) \in V^n$ $\quad B_{x_1,\ldots,x_n} \leq A^n$ $\qquad (x_1,\ldots,x_n) \in B_{x_1 \ldots x_n}$
proper

Lemma 1) $\forall$ fin. rel. str. $\mathbb{G} = (A; \mathcal{R})$ with $\mathcal{R}$ fin.
$\exists\, n \geq 1$ s.t.
$$CSP(\mathbb{G}) \equiv_L CSP_0(\mathrm{PolAlg}(\mathbb{G}), n)$$

Pf) Let $n = \max(\mathrm{arity}(R) : R \in \mathcal{R})$
Put $\mathcal{S}$ = the relations of $\mathbb{G}(\underset{\sim}{A}, n)$ where $\underset{\sim}{A} = \mathrm{PolAlg}(\mathbb{G})$

$$= \{ B : \underset{\sim}{B} \leq \underset{\sim}{A}^k \quad 1 \leq k \leq n \}$$

(So $CSP(\mathrm{PolAlg}(\mathbb{G}, n)) = CSP(\underset{\sim}{A}, n) = CSP((A; \mathcal{S}))$)

Note: $\mathcal{R} \subseteq \mathcal{S}$
Conversely, every $B \in \mathcal{S}$ is compatible with $\underset{\sim}{A} = \mathrm{PolAlg}(\mathbb{G})$, so B is pp-definable in $\mathbb{G}$.
Both $\mathcal{R}, \mathcal{S}$ are finite, so
$$CSP(\underbrace{(A; \mathcal{R})}_{\mathbb{G}}) \equiv_L CSP((A; \mathcal{S}))$$

Remains to reduce $CSP(\underset{\sim}{A}, n)$ to $CSP_0(\underset{\sim}{A}, n)$.

Let $(\underset{\text{variables}}{V}, \underset{\text{constraints}}{\Pi})$ be an instance of $CSP(\underset{\sim}{A}, n)$.

Build an equivalent instance of $CSP_0(\underset{\sim}{A}, n)$
Fix lin. order $<$ of V
while $\exists$ constraint $(\underbrace{(x_{i_1}, \ldots, x_{i_k})}_{\bar{x}}, R) \in \Pi$ with $\bar{x}$ improper

Case 1: $\bar{x} = (x_1, \ldots, x_s, \ldots, \underset{t}{x_s}, \ldots, x_k)$

Replace constraint with
$((x_1, \ldots, x_s, \ldots, x_t, x_{t+1}, \ldots, x_k), \overset{(k-1)\text{-ary}}{R^{s=t}} = \{(a_1, \ldots, a_{k-1}) : (a_1 \ldots a_s \ldots a_{t-1}, a_s, a_{t+1}) \in R\}$

Note: $R^{s=t} \in \mathcal{S}$

If $R^{s=t}$ is empty STOP answer NO.

Ultimately, all constraints in Π will have proper scope.
- If $\bar{x} \in V^k$, $k \le n$ is proper but not the scope of any constraint: add a new constraint $(\bar{x}; A^k)$

Finally, given a proper $\bar{x} \in V^k$ list all constraints having $\bar{x}$ as a scope
$(\bar{x}, R_1), (\bar{x}, R_2), \ldots, (\bar{x}, R_m)$
Replace with single constraint
$(\bar{x}, \bigcap_{i=1}^{m} R_i)$.

$\square$

Simplification: An instance of $CSP_0(\underset{\sim}{A}, n)$ can be written (disposing of $<$)

$$\left(V, \{(x, B_x) : x \in V\}, \{\{x, y\}, B_{xy} : \{x, y\} \subseteq V, x \ne y \ldots\right)$$

For $\emptyset \ne I \subseteq V$ $|I| \le n$, have (I, B_I)

Lemma 2| $\forall$ finite algebra $\underset{\sim}{A}$, $\forall n \ge 1$, $\exists$ fin. alg. $\underset{\sim}{D}$ s.t.
- $CSP(\underset{\sim}{A}, n) \le_L CSP_0(\underset{\sim}{D}, 2)$
- $\underset{\sim}{A}, \underset{\sim}{D}$ satisfy the same Maltsev conditions.

Pf| Let $\underset{\sim}{D} = \underset{\sim}{A}^{\lceil \frac{n}{2} \rceil}$

Given an instance for $CSP_0(\underset{\sim}{A}, n)$:

$$\left(V, (B_I)_{\substack{I \subseteq V \\ 1 \le |I| \le n}}\right)$$

Put $V^* = \{I : I \subseteq V, |I| = \lceil \frac{n}{2} \rceil\}$.

$\boxed{\text{Homework: Fix this.}}$

Unary constraints
For $I \in V^*$, $B^*_I = \cancel{B_I \le \underset{\sim}{A}^{|I|}} = \underset{\sim}{D}$

For $I, J \in V^*$, $B^*_{I,J} := \{(\bar{a}, \bar{b}) \in D \times D : \exists \bar{c} \in B_{I \cup J}$ s.t. $\bar{c}|_I = \bar{a}, \bar{c}|_J = \bar{b}\}$

<u>Claim</u>: $(V, (B_I)_I)$ has a solution $\Leftrightarrow$

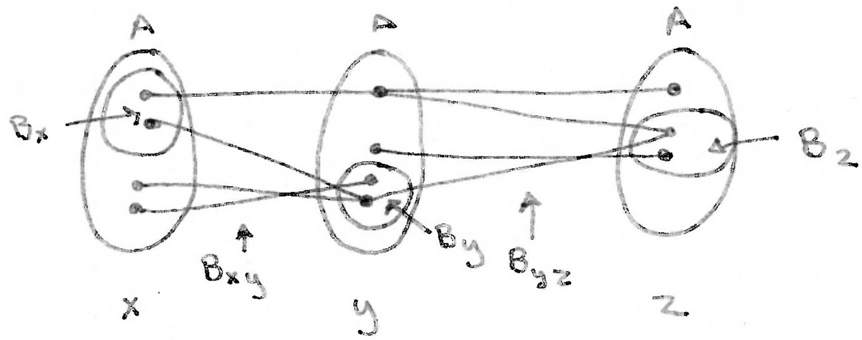
$(V^*, (B^*_I, B^*_{I,J}))$ has a solution

Now consider an instance of $CSP_0(\underset{\sim}{A}, 2)$

$(V, (B_x)_{x\in V}, (B_{xy})_{x,y\in V})$

variables

$\underset{\sim}{B}_x \leq \underset{\sim}{A}$ $\qquad$ $\underset{\sim}{B}_{xy} \leq \underset{\sim}{A}^2$

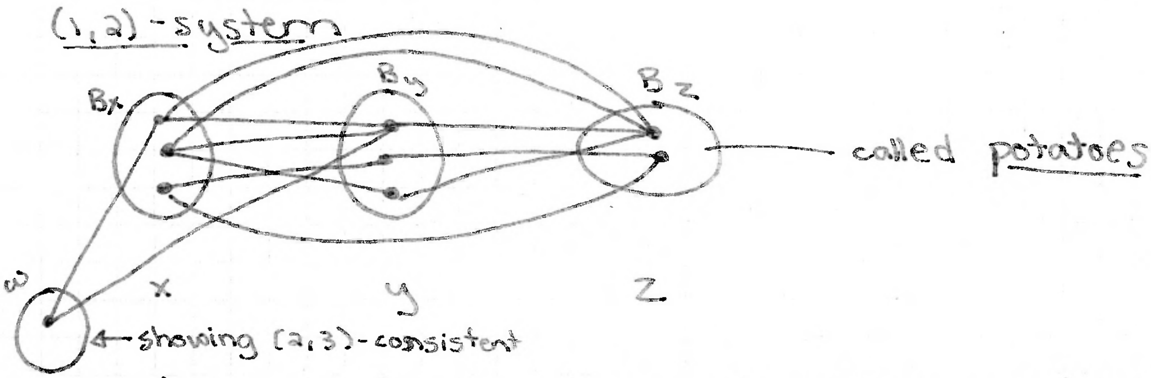


<u>Defn</u> Such an instance is a <u>(1,2)-system</u> if $\forall x, y$

(1) $B_{x,y} \subseteq B_x \times B_y$

(2) $\forall a \in B_x \ \exists b \in B_y$ with $(a,b) \in B_{x,y}$ and conversely $\forall b \in B_y \ \exists a \in B_x$ with $(a,b) \in B_{x,y}$

<u>(1,2)-system</u>



<u>Claim</u> The (1,2)-consistency algorithm reduces $CSP_0(\underset{\sim}{A}, 2)$ to (1,2)-system instances of $CSP_0(\underset{\sim}{A}, 2)$.

<u>Defn</u> A $(1,2)$-system is <u>$(2,3)$-consistent</u> if $\forall\, x, y, w \in V$ every edge in $B_{x,y}$ can be extended to B_w to a triangle $B_{xy} - B_{xw} - B_{yw}$

<u>$(2,4)$-consistency</u> every edge extends to a 4-clique.

For any fixed k, we can assume instances to $CSP_0(\underset{\sim}{A}, 2)$ are $(2,k)$-consistent $(1,2)$-systems.

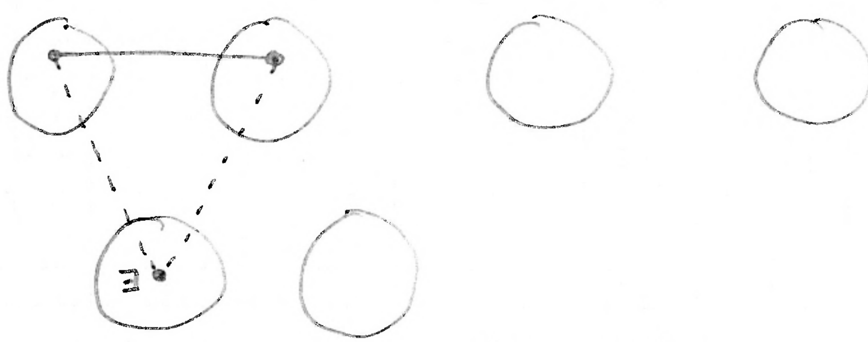
$CSP(\underset{\sim}{A}, 2)$ has <u>bounded width</u> if (but not only if) $\exists\, k$ s.t. every $(2,k)$-consistent system has a solution.

<u>Thm</u> (Barto, Kozik) Suppose $\underset{\sim}{A}$ omits types 1, 2. Then $CSP(\underset{\sim}{A}, 2)$ has bounded width. In fact, every $(2,3)$-consistent instance of $CSP_0(\underset{\sim}{A}, 2)$ has a solution.

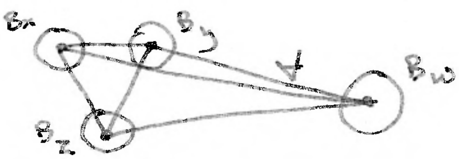
<u>Sketch of pf.</u> that if $\underset{\sim}{A}$ has a 3-ary NU operation

$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx x$

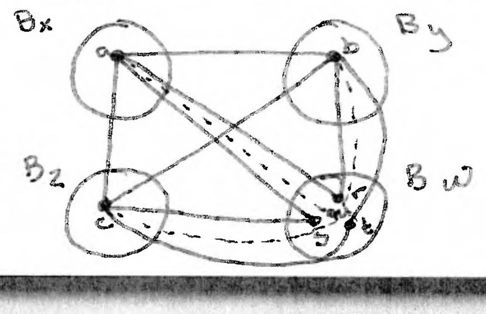
Consider a $(2,3)$-consistent instance



<u>Claim</u>: For Every triangle



can be extended to a 4-clique



by $(2,3)$-consistency every edge extends to a Δ through B_w
$\exists\, r$, $\exists\, s$, $\exists\, t$ in B_w

Let f be our 3-ary NU of $\underset{\sim}{A}$
f preserves all constraints.

Let $u = f(r, s, t)$. f preserves B_ω so $u \in B_\omega$

Claim: $(a, u) \in B_{x,\omega}$ Have $(a, r) \in B_{x,\omega}$
$(a, s) \in B_{x,\omega}$

By (1,2)-system some $p \in B_x$ s.t. $(p, t) \in B_{x,\omega}$

Apply f to these three

$$(f(a,a,p), f(r,s,t)) \in B_{x\omega}$$

Can do same to get (b, u) and (c, u) in $B_{x,\omega}$ to get 4-clique.