Weak multiplier Hopf algebras versus multiplier Hopf algebroids

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Outline of the talk

Outline:

This talk is about joint work in progress with Thomas Timmermann from the University of Muenster.

Introduction

Recall the definition of a groupoid.

Definition

A groupoid is a set G with a multiplication that is not necessarily everywhere defined. The product pq of two elements $p, q \in G$ is only defined when the so-called source $s(p)$ of p is the same as the target $t(q)$ of q. The source and target are maps from the groupoid to the set of units, denoted as G_0. Often this set is identified as a subset of G. We then have

$$ps(p) = p \qquad \text{and} \qquad t(p)p = p$$

for all $p \in G$. The multiplication is associative in the obvious sense. Moreover, for any element p in G there is an inverse p^{-1} satisfying $pp^{-1} = t(p)$ and $p^{-1}p = s(p)$. So we have e.g.

$$pp^{-1}p = p \qquad \text{and} \qquad p^{-1}pp^{-1} = p^{-1}.$$

Introduction

Some trivial examples

Here are the basic (trivial) examples.

Example

Any group is a groupoid. In this case $s(p) = t(p) = e$ for any element p where e is the identity in G. There is only one unit, namely the unit of the group.

Next, we have the other extreme.

Example

Take any set X and let $G = X \times X$. Define $qp = (z, x)$ if $q = (z, y)$ and $p = (y, x)$. Then G is a groupoid. The set G_0 of units is X, the source and the target of (y, x) are respectively x and y. The inverse of (y, x) is (x, y). The unit set G_0 is imbedded in G via the map $x \mapsto (x, x)$.

These examples are (in a way) special cases of the following.

Introduction

The action groupoid

Proposition

Let X be a set and H a group. Assume that H acts on X from the left. We use hx for the action of an element $h \in H$ on an element $x \in X$. Define

$$G = \{(y, h, x) \mid x, y \in X \text{ satisfying } y = hx\}.$$

Then G is a groupoid when the product qp of two elements $q = (z, k, y)$ and $p = (y', h, x)$ is only defined if $y = y'$ and then given by $qp = (z, k, y)(y, h, x) := (z, kh, x)$.

The set G_0 of units is X itself. We have $s(p) = x$ and $t(p) = y$ if $p = (y, h, x)$ and we have $p^{-1} = (x, h^{-1}, y)$. The units are identified as a subset of G by the map $x \mapsto (x, e, x)$ where e is the identity in the group H.

Introduction

The associated weak multiplier Hopf algebras

We start with a groupoid G. Let $K(G)$ be the algebra of complex functions with finite support on G and pointwise operations. We define a coproduct Δ on $K(G)$ by $\Delta(f)(p,q) = f(pq)$ for $p, q \in G$ if pq is defined and $\Delta(f)(p,q) = 0$ otherwise.

Proposition

The pair $(K(G), \Delta)$ is a weak multiplier Hopf algebra.

It is also possible to look at the dual. This is the groupoid algebra $\mathbb{C}G$ with the coproduct Δ defined by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all $p \in G$ where $p \mapsto \lambda_p$ denotes the canonical imbedding of G in $\mathbb{C}G$. Again the pair $(\mathbb{C}G, \Delta)$ is a weak multiplier Hopf algebra. It is the dual of $(K(G), \Delta)$.

Remark that $K(G)$ does not have an identity if G is infinite. Also $\mathbb{C}G$ will not be unital if the set of units is infinite.

Multiplier Hopf algebras

Recall the definition of a multiplier Hopf algebra.

Definition

Suppose that

- A is an algebra with a non-degenerate product,
- $\Delta : A \to M(A \otimes A)$ is a coproduct,
- the canonical maps T_1 and T_2, defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b)$$

  are bijective maps from $A \otimes A$ to $A \otimes A$.

Then (A, Δ) is a multiplier Hopf algebra.

For a weak multiplier Hopf algebra, the canonical maps are no longer assumed to be bijective.

Weak multiplier Hopf algebras

Definition (preliminary)

A pair (A, Δ) will be a weak multiplier Hopf algebra if:

- *A is an idempotent algebra with a non-degenerate product.*
- *$\Delta : A \to M(A \otimes A)$ is a full coproduct with a counit.*
- *There is multiplier $E \in M(A \otimes A)$ determining the ranges of the canonical maps T_1 and T_2 (playing the role of $\Delta(1)$).*
- *The kernels of the canonical maps are also determined by E in a specific way.*

Theorem

There is a unique antipode S giving 'generalized inverses' of the canonical maps. It is a linear map $S : A \to M(A)$ and it is both an anti-algebra and an anti-coalgebra map.

The main definition

Definition

A pair (A, Δ) will be a weak multiplier Hopf algebra if:

- A is an idempotent algebra with a non-degenerate product.
- $\Delta : A \to M(A \otimes A)$ is a full coproduct with a counit.
- There is an idempotent multiplier $E \in M(A \otimes A)$ so that

  $$\Delta(A)(1 \otimes A) = E(A \otimes A) \quad \text{and} \quad (A \otimes 1)\Delta(A) = (A \otimes A)E$$

  and

  $$(\iota \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1).$$
- The kernels of the canonical maps are given by the ranges of the idempotents $1 - F_1$ and $1 - F_2$ respectively where F_1 and F_2 are obtained as follows.

The main definition

The idempotent elements F_1 and F_2

Let (A, Δ) and E in $M(A \otimes A)$ be as before.

Proposition

There exists a right multiplier F_1 of $A \otimes A^{op}$ and a left multiplier F_2 of $A^{op} \otimes A$, uniquely determined by

$$E_{13}(F_1 \otimes 1) = E_{13}(1 \otimes E) \quad \text{and} \quad (1 \otimes F_2)E_{13} = (E \otimes 1)E_{13}.$$

Remark

- *These idempotents F_1 and F_2 define idempotent maps G_1 and G_2 from $A \otimes A$ to itself by*

$$G_1(a \otimes b) = (a \otimes 1)F_1(1 \otimes b)$$
$$G_2(a \otimes b) = (a \otimes 1)F_2(1 \otimes b).$$

- *We have $T_1 \circ (1 - G_1) = 0$ and $T_2 \circ (1 - G_2) = 0$.*

Existence of the antipode

Definition

A generalized inverse R_1 of T_1 is a linear map from $A \otimes A$ to itself so that $T_1 R_1 T_1 = T_1$ and $R_1 T_1 R_1 = R_1$. Similarly for T_2.

These generalized inverses are completely determined by a choice of projections on the ranges and on the kernels.

Proposition

There exists a unique linear map S from A to $M(A)$, such that the maps R_1 and R_2 given by

$$R_1(a \otimes b) = \sum_{(a)} a_{(1)} \otimes S(a_{(2)})b$$

$$R_2(a \otimes b) = \sum_{(b)} aS(b_{(1)}) \otimes b_{(2)}$$

are generalized inverses of the canonical maps T_1 and T_2.

Properties of the antipode

Remark

- *First, we obtain maps S_1 and S_2 giving R_1 and R_2 respectively. The fact that S_1 and S_2 actually coincide is a consequence of the formulas giving the idempotents F_1 and F_2 in terms of E. This is a remarkable fact.*
- *We have*

$$\sum_{(a)} a_{(1)} S(a_{(2)}) a_{(3)} = a$$

$$\sum_{(a)} S(a_{(1)}) a_{(2)} S(a_{(3)}) = S(a).$$

- *If the map S is bijective from A to itself, we call the weak multiplier Hopf algebra regular. This happens, as in the case of multiplier Hopf algebras, precisely if flipping the coproduct on A (or the multiplication in A) still yields a weak multiplier Hopf algebra.*

The source and target maps

Recall that in a groupoid, the product pq of two elements p, q is defined if the source $s(p)$ is equal to the target $t(p)$. They are thought of as elements in G and we have the formulas

$$s(p) = p^{-1}p \qquad \text{and} \qquad t(p) = pp^{-1}$$

for all $p \in G$.

Definition

Assume that (A, Δ) is a weak multiplier Hopf algebra with antipode S.The source and target maps ε_s and ε_t are defined as

$$\varepsilon_s(a) = \sum S(a_{(1)})a_{(2)} \qquad \text{and} \qquad \varepsilon_t(a) = \sum a_{(1)}S(a_{(2)}).$$

The source and target algebras

The source and target maps, map into the source and target algebras A_s and A_t. They are defined as follows.

Definition

Let E be the canonical idempotent of the weak multiplier Hopf algebra (A, Δ). Then we denote

$$A_s = \{y \in M(A) \mid \Delta(y) = E(1 \otimes y)\}.$$

$$A_t = \{x \in M(A) \mid \Delta(x) = (x \otimes 1)E\}.$$

Remark

- *The spaces $\varepsilon_s(A)$ and $\varepsilon_t(A)$ are subalgebras of A_s and A_t respectively. In fact, we can show that A_s and A_t are the multiplier algebras of $\varepsilon_s(A)$ and $\varepsilon_t(A)$.*
- *The algebras A_s and A_t (or rather $\varepsilon_s(A)$ and $\varepsilon_t(A)$) are the 'left' and the 'right' leg of E and $E \in M(\varepsilon_s(A) \otimes \varepsilon_t(A))$.*

The canonical maps between balanced tensor products - the map T_1

Consider the map T_1 from $A \otimes A$ to itself. We know that the range is $E(A \otimes A)$ and that the kernel is $(A \otimes 1)(1 - F_1)(1 \otimes A)$.

Proposition

Define $A \otimes_s A$ as the quotient of $A \otimes A$ by the subspace spanned by $ay \otimes a' - a \otimes ya'$ where $a, a' \in A$ and $y \in \varepsilon_s(A)$. Define $A \otimes_\ell A$ as the quotient of $A \otimes A$ by the subspace spanned by $ya \otimes a' - a \otimes S(y)a'$. The map T_1, defined from $A \otimes_s A$ to $A \otimes_\ell A$ is a bijection.

The proof is based on

- $\Delta(ay)(1 \otimes a') = \Delta(a)(1 \otimes ya')$ for $a, a' \in A$ and $y \in \varepsilon_s(A)$,
- $mF_1 = \sum E_{(1)}S(E_{(2)}) = 1$ and the left leg of F_1 is in $\varepsilon_s(A)$,
- $\sum S(E_{(1)})E_{(2)} = 1$.

The canonical maps between balanced tensor products - the map T_2

Consider the map T_2 from $A \otimes A$ to itself. We know that the range is $(A \otimes A)E$ and that the kernel is $(A \otimes 1)(1 - F_2)(1 \otimes A)$.

Proposition

Define $A \otimes_t A$ as the quotient of $A \otimes A$ by the subspace spanned by $ax \otimes a' - a \otimes xa'$ where $a, a' \in A$ and $x \in \varepsilon_t(A)$. Define $A \otimes_r A$ as the quotient of $A \otimes A$ by the subspace spanned by $aS(x) \otimes a' - a \otimes a'x$. The map T_2, defined from $A \otimes_t A$ to $A \otimes_r A$ is a bijection.

The proof is based on

- $(ax \otimes 1)\Delta(a') = (a \otimes 1)\Delta(xa')$ for $a, a' \in A$ and $x \in \varepsilon_t(A)$,
- $mF_2 = \sum S(E_{(1)})E_{(2)} = 1$ and the right leg of F_2 is in $\varepsilon_t(A)$,
- $\sum E_{(1)}S(E_{(2)}) = 1$.

The concept of a multiplier Hopf algebroid

The basic ingredients are a triple (A, B, C) where A is a non-degenerate idempotent algebra, B and C are commuting subalgebras, sitting nicely in $M(A)$, together with two anti-isomorphisms $S_B : B \to C$ and $S_C : C \to B$. Then the balanced tensor products $A \otimes_s A$, $A \otimes_\ell A$, $A \otimes_t A$ and $A \otimes_r A$ can be defined as before.

Now, a multiplier Hopf algebroid is, roughly speaking, given by a pair of coproducts Δ_B and Δ_C so that the associated maps T_1 and T_2 given by

$$T_1(a \otimes b) = \Delta_B(a)(1 \otimes c) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta_C(b)$$

are bijective between the appropriate balanced tensor products.

In other words, one forgets where these maps came from.

More precise definitions

Consider again the balanced tensor product $A \otimes_\ell A$. We have $ya \otimes a' = a \otimes S_B(y)a'$ in $A \otimes_\ell A$ when $y \in B$. We let A act from the right by multiplication in each of the two factors.

Notation

Denote by $A \overline{\otimes}_\ell A$ the extended module. Elements z in $A \overline{\otimes}_\ell A$ have the property (by definition) that

$$z(a \otimes 1) \qquad \text{and} \qquad z(1 \otimes a)$$

belong to $A \otimes_\ell A$ for all $a \in A$. Next we consider the subspace of elements z in $A \overline{\otimes}_\ell A$ satisfying

$$z(y \otimes 1) = z(1 \otimes S_B(y)) \qquad \text{for all } y \in B.$$

This subspace is an algebra and it is denoted as $L_{reg}(A_B \times A)$.

The left and the right coproducts

Definition

A left coproduct is a homomorphism $\Delta_B : A \to L_{\text{reg}}(A_B \times A)$ satisfying

$$\Delta_B(yay') = (1 \otimes y)\Delta_B(a)(1 \otimes y') \tag{1}$$
$$\Delta_B(xax') = (x \otimes 1)\Delta_B(a)(x' \otimes 1) \tag{2}$$

whenever $a \in A$, $y, y \in B$ and $x, x' \in C$.

Similarly a right coproduct is defined as a homomorphism from A to an algebra $L_{\text{reg}}(A \times_C A)$ sitting in the extended module of $A \otimes_r A$. We have the associated canonical maps

$$T_1(a \otimes a') = \Delta_B(a)(1 \otimes a') \quad \text{and} \quad T_2(a \otimes a') = (a \otimes 1)\Delta_C(a').$$

They are maps from $A \otimes_s A$ to $A \otimes_\ell A$ and from $A \otimes_t A$ to $A \otimes_r A$ respectively. They are assumed to be bijective.

Coassociativity of the coproducts

First there are the assumptions of coassociativity of the left and the right coproduct. For the left coproduct, it is expressed as

$$(\Delta_B \otimes \iota)(\Delta_B(a)(1 \otimes c'))(c \otimes 1 \otimes 1) = (\iota \otimes \Delta_B)(\Delta_B(a)(c \otimes 1))(1 \otimes 1 \otimes c')$$

One needs a form of regularity of Δ_B and furthermore, one has to check that all these maps are well-defined on the various balanced tensor products!

We have a similar form of coassociativity of the right coproduct Δ_C.

We also have to relate the two coproducts, but we can not say that they are equal as they map to different spaces. Instead, we have another form of coassociativity

$$(c \otimes 1 \otimes 1)(\Delta_C \otimes \iota)(\Delta_B(a)(1 \otimes c')) = (\iota \otimes \Delta_B)((c \otimes 1)\Delta_C(a))(1 \otimes 1 \otimes c')$$

The counital maps

On a regular multiplier Hopf algebroid, we have a left and a right counit.

Definition

A left counit is a linear map $\varepsilon_B : A \to B$ such that

$$\varepsilon_B(ya) = y\varepsilon_B(a) \quad \text{and} \quad \varepsilon_B(S(y)a) = \varepsilon_B(a)y$$

and so that

$$(\varepsilon_B \otimes \iota)(\Delta_B(a)(1 \otimes c)) = ac$$

with the identification $B \otimes A \to A$ given by $y \otimes a \mapsto S(y)a$.

One again has to be careful and verify that the maps behave properly with respect to the module actions.

Similarly a right counit is defined.

The antipode

Definition

An antipode is an anti-isomorphism from A to A. It has to coincide with the maps S_B and S_C given on B and on C resp. And it satisfies the expected formulas

$$m(\iota \otimes S)((c \otimes 1)\Delta_C(a)) = cS_B(\varepsilon_B(a)) \tag{3}$$

$$m(S \otimes \iota)((\Delta_B(a)(1 \otimes c)) = S_C(\varepsilon_C(a))c \tag{4}$$

And here again, one has to verify that the maps and formulas are compatible with the various module actions.

An example

Let B and C be two non-degenerate and idempotent algebras. Assume that $S_B : B \to C$ and $S_C : C \to B$ are anti-isomorphisms.

Proposition

Define $A = C \otimes B$ and identify B and C as subalgebras of $M(A)$. Then define $\Delta_B : A \to A\overline{\otimes}_\ell A$ by $\Delta_B(cb) = c \otimes b$ for $b \in B$ and $c \in C$. Similarly, define $\Delta_C : A \to A\overline{\otimes}_r A$. Then $(A, B, C, S_B, S_C, \Delta_B, \Delta_C)$ is a multiplier Hopf algebroid. The counital maps are given by

$$\varepsilon_B(bc) = bS_B^{-1}(c) \qquad \text{and} \qquad \varepsilon_C(bc) = S_C^{-1}(b)c.$$

The antipode is given by $S(cb) = S_B(b)S_C(c)$.

The proof is straightforward.

An example

If this multiplier Hopf algebroid comes from a weak multiplier Hopf algebra, there will be an idempotent $E \in M(B \otimes C)$ and the coproduct Δ on $C \otimes B$ will be given by $\Delta(c \otimes b) = c \otimes E \otimes b$. It will also follow that the underlying algebras B and C are *separable Frobenius*. In particular, there will be a faithful linear functional on B.

Therefore, if we want to find an example of a multiplier Hopf algebroid, not coming from a weak multiplier Hopf algebra, we just have to find an algebra B with no faithful functional. We then can take for C the algebra B with the opposite product. This turns out to be possible, even for algebras with an identity. Remember that a unital algebra is automatically idempotent and non-degenerate.

Example

Consider any vector space V *and make it into an algebra by defining the product of any two elements equal to* 0. *Let* $B = \widetilde{V}$, *the algebra obtained by adding an identity. So, any element in* B *is of the form* $v + \lambda 1$ *for* $v \in V$ *and* $\lambda \in \mathbb{C}$. *And the product of two elements is given as*

$$(v + \lambda 1)(w + \mu 1) = \mu v + \lambda w + (\lambda\mu)1.$$

Any linear functional on B *is of the form* $v + \lambda 1 \mapsto f(v) + t\lambda$ *where* f *is a linear functional on* V *and* t *a complex number. If now* $a = v$ *with* $f(v) = 0$, *then* $f(ab) = 0$ *for all* $b \in B$. *Hence there is no faithful linear functional on* B. *Because the algebra has an identity, it is non-degenerate and idempotent. If we take for* C *the opposite algebra, and for* S_B *and* S_C *the identity maps, we can construct a multiplier Hopf algebroid. It will not come from a weak multiplier Hopf algebra.*

Conclusions

- First we have the notion of a (multiplier) Hopf algebra. Any group gives rise to a dual pair of multiplier Hopf algebras. If the group is finite, we have Hopf algebras.
- Next there is the notion of a weak (multiplier) Hopf algebra. Any groupoid gives a dual pair of weak multiplier Hopf algebras. If the groupoid is finite, we have weak Hopf algebras.
- Passing to view the canonical maps between balanced tensor products, we arrive at the notion of a (multiplier) Hopf algebroid.
- This is a more general theory, better suited as a concept of a quantum groupoid.
- The case of a multiplier Hopf *-algebroid with positive integrals should provide a link with the measured quantum groupoids.

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- *Weak multiplier Hopf algebras I. The main theory*.
- *Weak multiplier Hopf algebras II. The source and target algebras*.
- *Weak multiplier Hopf algebras III. Integrals and duality*.