Conformal removability of Schramm Loewner Evolutions

Konstantinos Kavvadias

joint with Jason Miller and Lukas Schoug

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Conformal welding

Suppose that Γ is a closed Jordan curve in the complex plane $\mathbf{C}$ and let U (resp. $\widetilde{U}$) be the bounded (resp. unbounded) connected component of $\mathbf{C} \setminus \Gamma$. Let $f \colon \mathbf{D} \to U$ and $g \colon \mathbf{C} \setminus \overline{\mathbf{D}} \to \widetilde{U}$ be conformal transformations. By Caratheodory's theorem, f and g extend to homeomorphisms $\widetilde{f} \colon \partial\mathbf{D} \to \Gamma$ and $\widetilde{g} \colon \partial\mathbf{D} \to \Gamma$. Then $h = \widetilde{g}^{-1} \circ \widetilde{f} \colon \partial\mathbf{D} \to \partial\mathbf{D}$ is a homeomorphism and homeomorphisms arising in this way are called *conformal weldings*.

Uniqueness of the welding interface

- Fix a welding homeomorphism $\phi\colon \partial\mathbf{D} \to \partial\mathbf{D}$ with two welding interfaces η_1, η_2 and corresponding pairs of conformal maps (f_1, g_1) and (f_2, g_2).

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- Let U_j (resp. $\widetilde{U}_j$) be the bounded (resp. unbounded) connected component of $\mathbf{C} \setminus \eta_j$ for $j = 1, 2$. Then $f_2 \circ f_1^{-1}$ is a conformal transformation mapping U_1 onto U_2 and $g_2 \circ g_1^{-1}$ is a conformal transformation mapping $\widetilde{U}_1$ onto $\widetilde{U}_2$.

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- We can define a homeomorphism $\psi\colon \mathbf{C} \to \mathbf{C}$ by setting $\psi(z) = f_2 \circ f_1^{-1}(z)$ for $z \in U_1$, $\psi(z) = g_2 \circ g_1^{-1}(z)$ for $z \in \widetilde{U}_1$ and $\psi(z) = g_2 \circ g_1^{-1}(z) = f_2 \circ f_1^{-1}(z)$ for $z \in \eta_1$.

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- Suppose that η_1 is conformally removable. Then ψ is a Mobius transformation since it is conformal on $\mathbf{C} \setminus \eta_1$ and $\eta_2 = \psi(\eta_1)$.

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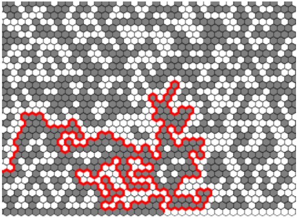

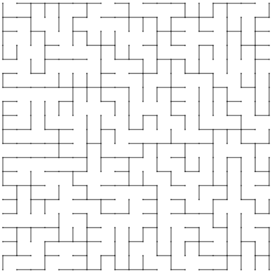

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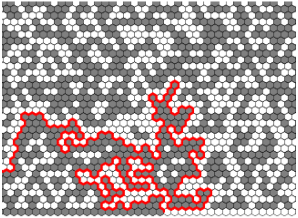

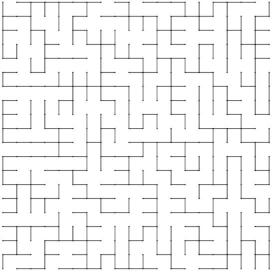

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- Conformally invariant.
- Three phases: curves are simple if $\kappa \in (0, 4]$, self-intersecting if $\kappa \in (4, 8)$ and space-filling if $\kappa \geq 8$.

Figure: SLE_κ for $\kappa = 2, 3, 6$. (Simulation by Tom Kennedy.)

Random conformal welding

- Fix $\kappa \in (0,4]$ and suppose that η is an SLE_κ curve in the upper half-plane $\mathbf{H}$ from 0 to ∞. **Question**: Is it possible to find a (random) homeomorphism $\phi\colon \mathbf{R}_+ \to \mathbf{R}_-$ so that the following holds?

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- Let D_L (resp. D_R) be the connected component of $\mathbf{H} \setminus \eta$ lying to the left (resp. right) of η. Let also ψ_L (resp. ψ_R) be a conformal transformation mapping $\mathbf{H}$ onto $\mathbf{H}_L$ (resp. $\mathbf{H}_R$) and fixing 0 and ∞. Then we need that $\phi = \psi_R^{-1} \circ \psi_L$.

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- Conformal removability of SLE_κ would imply that the only possible welding interface corresponding to ϕ is a rescaling of η in $\mathbf{H}$.

LQG random surface

- Let $\gamma \in (0,2]$. A γ-LQG surface is an equivalence class of pairs (D,h), where $D \subseteq \mathbf{C}$ is a simply connected domain and $h \in H^{-1}_{\text{loc}}(D)$ is a distribution on D. Two pairs (D_1,h_1) and (D_2,h_2) are defined to be equivalent if there exists a conformal map $\psi\colon D_2 \to D_1$ such that

$$h_2 = h_1 \circ \psi + Q_\gamma \log|\psi'| \tag{0.1}$$

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- For $\gamma \in (0, 2)$ and a γ-LQG surface (D, h) we can define Borel measures μ_h^γ and ν_h^γ on D and ∂D respectively via the regularization procedures

$$\mu_h^\gamma(dz) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz, \quad \nu_h^\gamma(dx) = \lim_{\epsilon \to 0} \epsilon^{\gamma^2/4} e^{\gamma h_\epsilon(x)/2} dx.$$

Solution of the random conformal welding problem

- Fix $\kappa = \gamma^2 \in (0,4)$ and let $(\mathbf{H}, h_L, 0, \infty)$, $(\mathbf{H}, h_R, 0, \infty)$ be two independent γ-quantum wedges. We let $\phi \colon \mathbf{R}_+ \to \mathbf{R}_-$ be the homeomorphism so that $\nu_{h_L}([0,x]) = \nu_{h_R}([\phi(x),0])$ for each $x \geq 0$. Then we can find an interface η in $\mathbf{H}$ from 0 to ∞ which is measurable with respect to (h_L, h_R) and such that $\phi = \psi_R^{-1} \circ \psi_L$, where ψ_q is the conformal transformation which maps $\mathbf{H}$ onto D_q and fixes 0 and ∞ for $q \in \{L, R\}$.

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- Conversely, suppose that we start with an $(\gamma - \frac{2}{\gamma})$-quantum wedge $(\mathbf{H}, h, 0, \infty)$ and let η be an SLE_κ in $\mathbf{H}$ from 0 to ∞ which is independent of η. Then the surfaces $(D_L, h|_{D_L}), (D_R, h|_{D_R})$ are independent γ-quantum wedges and their quantum boundary lengths along η agree.

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- The $\kappa < 4$ case was proved by Sheffield and the $\kappa = 4$ case by Holden and Powell.

Why "quantum wedges" are the natural surfaces to relate to SLE curves?

An a-quantum wedge $(\mathbf{H}, h, 0, \infty)$ has the following properties:

- $(\mathbf{H}, h, 0, \infty)$ and $(\mathbf{H}, h + C, 0, \infty)$ have the same law for every fixed $C > 0$ when viewed modulo the coordinate change formula (0.1). Note that the law of SLE_κ is scale invariant.

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- $(\mathbf{H}, h, 0, \infty)$ looks like $\widetilde{h} - a \log|\cdot|$ in arbitrarily small neighbourhoods of 0, where $\widetilde{h}$ is a free boundary GFF on $\mathbf{H}$ normalized so that its average on $\mathbf{H} \cap \partial\mathbf{D}$ is equal to 0.

How does one prove conformal removability?

Jones and Smirnov proved that if $K \subseteq \mathbf{C}$ is the boundary of a Hölder domain, then K is conformally removable. Rohde and Schramm proved that the complement of an SLE_κ in $\mathbf{H}$ for $\kappa < 4$ is a Hölder domain, so SLE_κ is conformally removable for $\kappa < 4$. More generally, Jones and Smirnov proved that if the uniformizing map has modulus of continuity $\exp(-\sqrt{\log(\delta^{-1})(\log(\log(\delta^{-1})))}/o(1))$ as $\delta \to 0$, then conformal removability of the boundary of the domain holds.

Why conformal removability of SLE_4 is hard?

The modulus of continuity of the SLE_4 uniformizing map is given by $(\log(\delta^{-1}))^{-\frac{1}{3}+o(1)}$ as $\delta \to 0$. The main reason for this is that SLE_4 curves are barely non-self-intersecting in the sense that they contain tight bottlenecks. Equivalently, if $z \notin \eta$ is such that $\mathrm{dist}(z, \eta) \asymp \epsilon$, then the probability that a complex Brownian motion independent of η and started at z travels macroscopic distance before hitting η behaves like $\exp(-\epsilon^{-3+o(1)})$ as $\epsilon \to 0$.

Quasiconformal maps

- Let $D, \widetilde{D}$ be domains in $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and let $f \colon D \to \widetilde{D}$ be an orientation preserving homeomorphism. We say that f is ACL (absolutely continuous on lines) if f is absolutely continuous on Lebesgue a.e. line segment in D which is parallel to one of the axes.

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- For $M \geq 1$, we say that f is an M-quasiconformal mapping if f is ACL and $\left|\frac{\partial f}{\partial \overline{z}}\right| \leq \left(\frac{M-1}{M+1}\right) \left|\frac{\partial f}{\partial z}\right|$ a.e. where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

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- f is conformal if and only if it is 1-quasiconformal. Moreover, for an M-quasiconformal map f we have that

$$\limsup_{r \to 0} \frac{M(z,r)}{m(z,r)} \leq M,$$

where $m(z,r) = \inf_{|w-z|=r} |f(w) - f(z)|$, $M(z,r) = \sup_{|w-z|=r} |f(w) - f(z)|$.

How do we show that $X \subseteq \mathbf{C}$ is conformally removable?

- Suppose that $\text{Leb}(X) = 0$ and $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism such that $f|_{\mathbf{C} \setminus X}$ is conformal. Since f is conformal a.e., it suffices to show that f has the ACL property.

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- Fix a rectangle $F = [a, b] \times [c, d]$ and sample t from $\text{Leb}([c, d])$. Set $L_t = \{x + it : x \in \mathbf{R}\}$. We need to show that $f|_{L_t \cap F}$ is absolutely continuous.

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- To show the latter, we need to control the variation of f near $L_t \cap X \cap F$.

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- There exists a path γ in A disconnecting $\partial^{\text{in}} A$ from $\partial^{\text{out}} A$ such that

$$\operatorname{diam}(f(\gamma)) \lesssim 2^{-(1-3a)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(w)|^2 dw. \tag{0.2}$$

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- We divide $L_t \cap F$ into intervals (I_j) of length 2^{-n} and surround each I_j by an annulus. Number of such intervals is $O(2^{(1-5a)n})$ and Fubini's theorem implies that $\int_{B(L_t \cap F, 2^{-(1-a^2)n})} |f'(w)|^2 dw \lesssim 2^{-(1-a^2)n}$.

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- Therefore the total variation of f near $L_t \cap X \cap F$ is at most

$$\sum_j \operatorname{diam}(f(\gamma_j)) \lesssim 2^{(1-5a)n}2^{-(1-a^2)(1-3a)n} + 2^{(1-a)n}\sum_j \int_{A_j\setminus X} |f'(w)|^2 dw$$

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Hyperbolic distance

- The hyperbolic metric in the unit disk $\mathbf{D}$ is defined by

$$\mathrm{dist}^{\mathbf{D}}_{\mathrm{hyp}}(z_1, z_2) = \inf\left\{\int_{z_1}^{z_2} \frac{|dz|}{1-|z|^2}\right\} \text{ for } z_1, z_2 \in \mathbf{D},$$

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  where the infimum is taken over all smooth curves in $\mathbf{D}$ connecting z_1 and z_2.
- It is conformally invariant in the sense that $\mathrm{dist}_{\mathrm{hyp}}^{\mathbf{D}}(\phi(z_1), \phi(z_2)) = \mathrm{dist}_{\mathrm{hyp}}^{\mathbf{D}}(z_1, z_2)$ for each conformal automorphism ϕ of $\mathbf{D}$. Hence we can define the hyperbolic distance $\mathrm{dist}_{\mathrm{hyp}}^{D}$ on a simply connected domain $D \subseteq \mathbf{C}$ by $\mathrm{dist}_{\mathrm{hyp}}^{D}(z_1, z_2) = \mathrm{dist}_{\mathrm{hyp}}^{\mathbf{D}}(\phi(z_1), \phi(z_2))$, where $\phi \colon D \to \mathbf{D}$ is a conformal transformation.

Whitney square decomposition

For any open subset $U \subseteq \mathbf{C}$, there exists a family $\mathcal{W} = (Q_j)$ of closed squares with pairwise disjoint interiors and sides parallel to the axes, so that Q_j has sidelength 2^{-n_j} for some $n_j \in \mathbf{Z}$, $U = \cup_j Q_j$ and such that

$$\operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j, \partial U) < 4\operatorname{diam}(Q_j).$$

Such a family is referred to as a Whitney square decomposition of U.

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- There exist simply connected subsets of $A \setminus X$, $U_1, \cdots, U_m$, segments $I_i \subseteq \partial U_{i-1} \cap \partial U_i$, and measures μ_i on I_i such that the following hold.
  (i) There exists $d_i \in (10a, 2-10a)$ such that $\mu_i(I_i) \geq M^{-1} 2^{-d_i k}$.
  (ii) $\mu_i(Y) \leq M \mathrm{diam}(Y)^{d_i - a}$ for every $Y \subseteq I_i$ Borel set.
  (iii) If $\mathcal{W}_i$ is a Whitney square decomposition of U_i, then there exists $z_i \in U_i$ such that for a large fraction of points $w \in I_i$ with respect to μ_i we have that $\text{disthyp}^{U_i}(z_i, Q) \leq M(2^k \text{length}(Q))^{-a}$ for every $Q \in \mathcal{W}_i$ such that $\gamma_{z_i,w}^{U_i} \cap Q \neq \emptyset$. Call those points "good".
  (iv) Number of $Q \in \mathcal{W}_i$ such that $\text{length}(Q) = 2^{-j}$ and $\gamma_{z_i,w}^{U_i} \cap Q \neq \emptyset$ for some $w \in I_i$ "good" point is at most $M 2^{(d_i + a)(i-k)}$.

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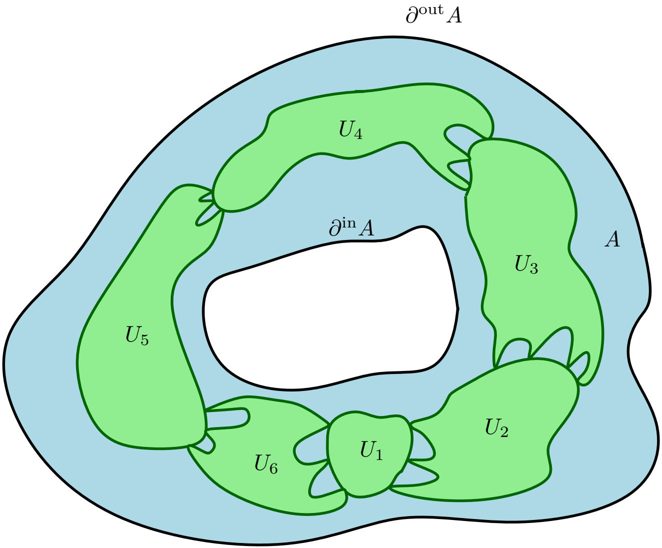
  - (iv) Number of $Q \in \mathcal{W}_i$ such that $\mathrm{length}(Q) = 2^{-j}$ and $\gamma^{U_i}_{z_i,w} \cap Q \neq \emptyset$ for some $w \in I_i$ "good" point is at most $M 2^{(d_i+a)(i-k)}$.
- Pick $w_i \in \partial U_{i-1} \cap \partial U_i$ "good" point and concatenate $\gamma^{U_i}_{z_i,w_i}, \gamma^{U_{i-1}}_{z_{i-1},w_i}$ for $2 \leq i \leq m$ to obtain γ.

Conformal removability condition



Conformal removability of SLE_4

- Our goal is to show that η (SLE_4) satisfies the conditions a.s. for every $K \subseteq \mathbf{H}$ compact. We are going to use the coupling of η with a GFF h on $\mathbf{H}$ where η is heuristically interpreted as the level set $\{x : h(x) = 0\}$. η is a measurable function of h under this coupling.

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- It is conjectured to be the time parameterization which arises when considering SLE as the scaling limit of an interface of a discrete model in which the curve is parameterized by the number of edges it crosses.

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- **Strategy**: Fix $z \in \mathbf{H}$, $k \in \mathbf{N}$ and $A_{z,k} = B(z, 2^{-k}) \setminus \overline{B(z, 2^{-k-1})}$. The event that $\eta|_{A_{z,k}}$ satisfies the desired conditions is determined by $h|_{A_{z,k}}$. The laws of the restrictions of h to disjoint annuli are approximately independent. So by applying a Borel-Cantelli type argument for a grid of points, it suffices to show that η satisfies the desired properties with arbitrarily high probability (by adjusting the parameters).

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- i and ii come from properties of natural parameterization.
- iv comes from the following observation. Fix $R > 0$ and for $\ell \in \mathbf{N}$, let N_ℓ be the number of points $z \in (2^{-\ell}\mathbf{Z})^2 \cap \mathbf{H} \cap B(0, R)$ such that $\eta \cap B(z, c2^{-\ell}) \neq \emptyset$, $c > 0$ constant. Then, $\mathbf{E}[N_\ell] \lesssim 2^{3\ell/2}$ and so $N_\ell \leq 2^{(3/2+a)\ell}$ for all ℓ sufficiently large a.s.

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- $\eta_0, \eta_1, \cdots, \eta_n, \eta_{n+1}$ crossings of $A_{z,k}$ made by η, where η_0 (resp. η_{n+1}) is the right (resp. left) side of the first crossing.

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- Sampling $\widetilde{\eta}$: first sample a whole-plane radial $\mathrm{SLE}_4(2)$ η_1 from ∞ to 0, and given η_1, sample an SLE_4 η_2 from 0 to ∞ in $\mathbf{C} \setminus \eta_1$.

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- Fix $H_0 > 0, \xi \in (0, H_0/10)$ small and suppose that $H_{z,k} \geq H_0$. Let $\widetilde{\eta}_1, \cdots, \widetilde{\eta}_n$ be the parts of $\varphi_{z,k}(\eta_1), \cdots, \varphi_{z,k}(\eta_n)$ in $(0,1) \times (3\xi, H_0 - 3\xi)$.

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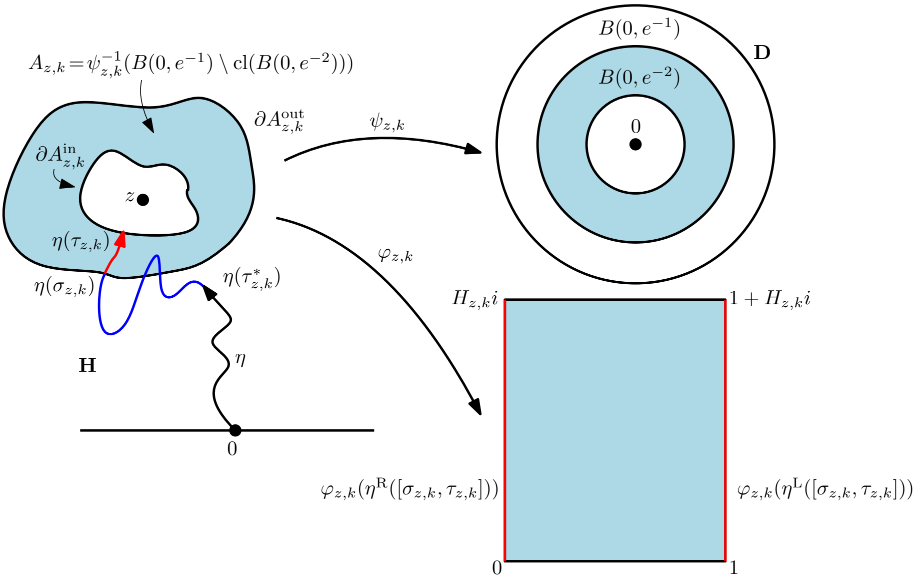

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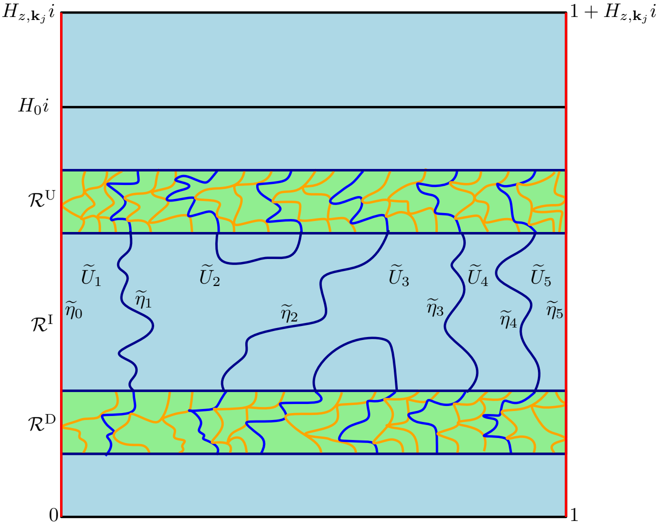
- Next step: do the same for $\widehat{h}$ to obtain the crossings $\widehat{\eta}_1, \cdots, \widehat{\eta}_n$. Then, show that with high probability the desired conditions hold for $\widehat{\eta}_1, \cdots, \widehat{\eta}_n$.

Conformal removability of SLE_4



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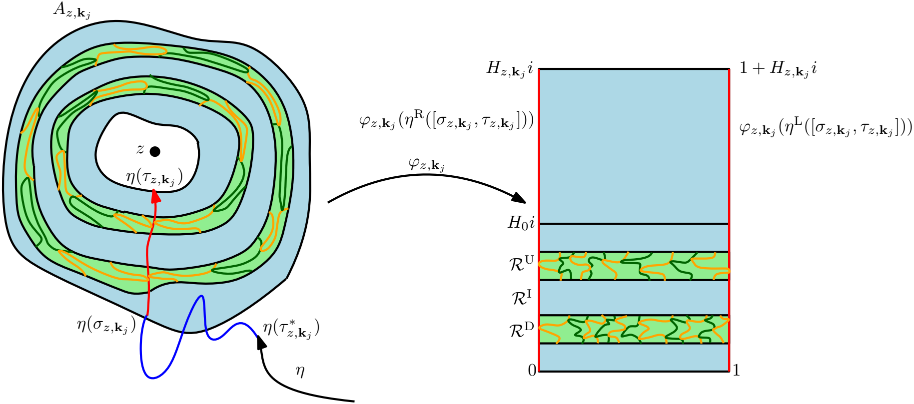

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- Finally, show that the $k_j(z)$'s are dense in the following sense: a.s. for every compact set $K \subseteq \mathbf{H}$, there exists $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$ and every $z \in (e^{-5n}\mathbf{Z})^2 \cap K$, if $\tau_{z,n} < \infty$, there exists $(1-a^2)n \leq k_j(z) \leq n$ and the desired properties hold for $A_{z,k_j(z)}$.

Explorartion

- Next we explore the crossings in a way which is measurable with respect to the fields. Fix $0 < a < b < H$ and set $\mathcal{R} = (0,1) \times (0,H), \mathcal{R}_{a,b} = (0,1) \times (a,b)$.

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- η_1 exits $\mathcal{R}_{a,b}$ in $\partial^{\mathsf{D}}\mathcal{R}_{a,b}$. Then, we explore the level line $\widetilde{\eta}_1$ of $-h + u$ from the point of $\partial^{\mathsf{L}}\mathcal{R}_{a,b}$ which has the largest imaginary part among the points visited by the exploration.
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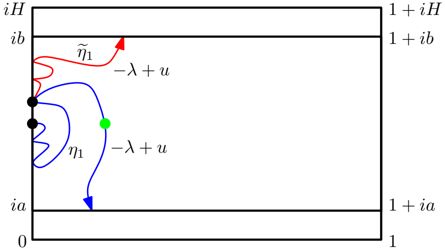

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- K_1: set discovered by the first stage of the exploration. Let L_1 be the rightmost crossing of K_1. Boundary values of h on L_1 are either $-\lambda + u$ or $\lambda + u$. We say that L_1 is a crossing of height u.

Exploration

- Suppose that we have defined the exploration after j steps without discovering level lines which hit $\partial^{\mathrm{R}}\mathcal{R}_{a,b}$. K_j: set discovered up until the j-th step, L_j: j-th crossing and u_j: height of L_j.

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- Repeat the step of the $j = 1$ case, replacing $\partial^{\mathrm{L}}\mathcal{R}_{a,b}$ with L_j and η_1 by η_{j+1} as follows.
  (i) If $u_{j+1} \notin (-2\lambda, 0)$, then η_{j+1} a.s. does not hit $\partial^{\mathrm{R}}\mathcal{R}_{a,b}$, so we proceed as before.
  (ii) If $u_{j+1} \in (-2\lambda, 0)$, then η_{j+1} or any subsequent level line might exit $\mathcal{R}_{a,b}$ in $\partial^{\mathrm{R}}\mathcal{R}_{a,b}$ and so we stop the exploration at that point.

Exploration

- Obtain crossings (L_j) of $\mathcal{R}_{a,b}$ such that $L_j \cap L_{j+1} \neq \emptyset$ for every j.

Exploration

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- A.s. the exploration discovers finitely many crossings (non-trivial).

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- SLE_κ curves for $\kappa \in (4, 8)$ arise as the welding interface where we glue together two independent stable looptrees where each loop is filled with a conditionally independent random surface (quantum disk).
- A stable looptree is a random space that comes equipped with a topology and some additional structure, e.g., each loop comes with a defined boundary length measure, and the looptree is a geodesic metric space.

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- We can think of an SLE_κ for $\kappa \in (4,8)$ as a random analogue of the Sierpinski gasket in the sense that the boundaries of its complementary connected components intersect each other.
- However, two distinct complementary connected components with non-empty intersection of the latter intersect at exactly one place whereas in the case of the former the intersection set is uncountable and has Hausdorff dimension $3 - \frac{3\kappa}{8}$.

Main result

- Suppose that $D \subseteq \mathbf{C}$ is open and $K \subseteq \mathbf{C}$ closed in D. We say that the adjacency graph of connected components of K in D is connected if for every pair of connected components U, V of $D \setminus K$, there exist connected components $U_1, \cdots, U_n$ of $D \setminus K$ such that $U = U_1, V = U_n$, and $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for every $1 \leq i \leq n-1$. Let $\mathcal{K}$ be the set of $\kappa \in (4, 8)$ such that the adjacency graph of complementary connected components of an SLE_κ in $\mathbf{H}$ is a.s. connected (Gwynne and Pfeffer proved that $\mathcal{K} \neq \emptyset$).

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- **Main result**: Fix $\kappa' \in \mathcal{K}$ and suppose that η' is an $\mathrm{SLE}_{\kappa'}$ in $\mathbf{H}$ from 0 to ∞. It a.s. holds that the range of η' is a.s. conformally removable.
- As in the $\kappa = 4$ case, we will couple η' with a GFF h on $\mathbf{H}$ so that η' is the counterflow line of h in $\mathbf{H}$ from 0 to ∞. Then, we will show that a.s., the conformal removability condition holds for η' at a sufficiently dense set of scales.

Construction of the chain of components

- **Main local connectivity statement**: Suppose that $\kappa' \in \mathcal{K}$ and fix $p \in (0,1)$ and an annulus $A \subseteq \mathbf{H}$ of size 2^{-k}. Then, with probability at least p, we can find $M \in \mathbf{N}$ and connected components $U_1, \cdots, U_n$ of $A \setminus \eta'$ with $n \leq M$, so that $\partial U_{j-1} \cap \partial U_j \neq \emptyset$, for every $1 \leq j \leq n$ (with the convention that $U_0 = U_n$ and $U_1 = U_{n+1}$), and the following hold.

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- (i) The upper Minkowski dimension of $\partial U_{j-1} \cap \partial U_j$ is at most $d_{\kappa'}^{\text{cut}} = 3 - \frac{3\kappa'}{8}$.
  (ii) There exists $\alpha = \alpha_{\kappa'} \in (0,1)$ such that the following is true. For each conformal transformation $\phi \colon \mathbf{D} \to O$, mapping $\mathbf{D}$ onto O with $O \in \{U_{j-1}, U_j\}$, there exists an open set $W \subseteq \mathbf{D}$ which contains a neighbourhood of $\phi^{-1}(\partial U_{j-1} \cap \partial U_j)$ in $\mathbf{D}$ such that $\phi|_W$ is Hölder continuous with exponent α.

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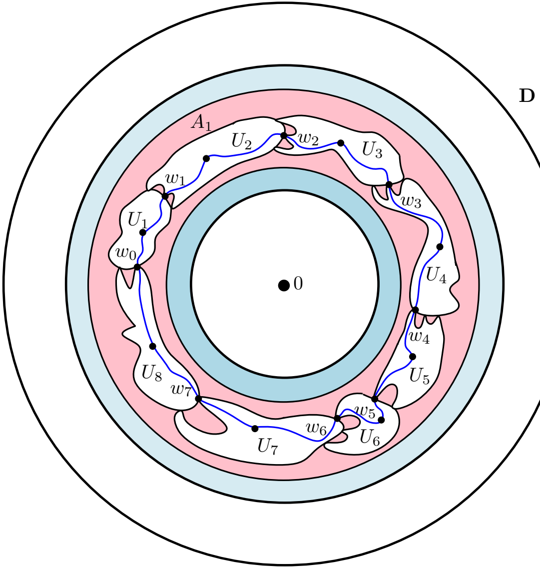
- The upper bound on the Minkowski dimension of $\partial U_{j-1} \cap \partial U_j$ follows from the fact that the points on $\partial U_{j-1} \cap \partial U_j$ are cut points of certain $\mathrm{SLE}_{\kappa'}$ curves and the later set has Minkowski dimension equal to $d_{\kappa'}^{\text{cut}}$. By cut points, we mean points of intersection of the left and right outer boundaries.

Construction of the chain of components



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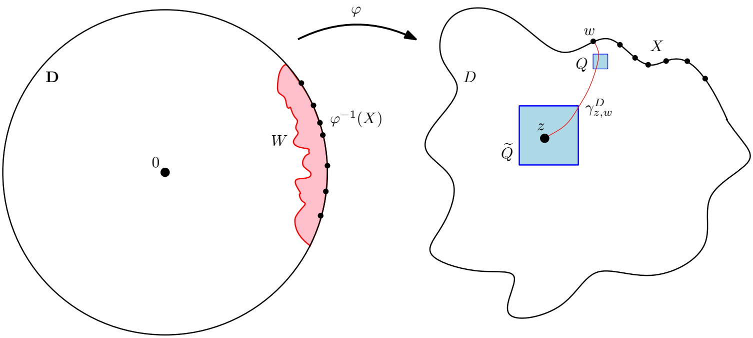

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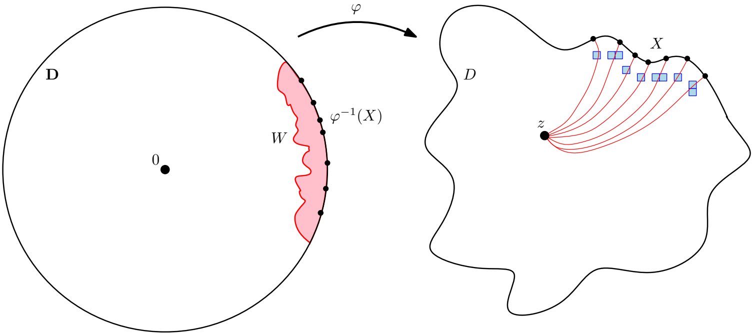

Construction of the measures

It remains to construct the measures μ_j and prove (i) and (ii). Since the events are locally determined by the underlying GFF, it suffices to assume that h is a GFF on $\mathbf{D}$ with boundary conditions so that the counterflow line η' of h has the law of a chordal $\mathrm{SLE}_{\kappa'}(\kappa' - 6)$ in $\mathbf{D}$ from $-i$ to i with the force point located at $(-i)^+$.

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- We continue the curve $\eta'|_{[t,\tau_t]}$ by targeting it at X_t. Let η'_t be the resulting curve and $\overline{\eta}'_t$ be its time-reversal. Then, we pick rational times $t, \overline{t} \in \mathbf{Q}_+$ such that the points on $\partial U \cap \partial V$ are cut points of the restriction of η' in $\mathbf{D} \setminus (\eta'([0,t]) \cup \overline{\eta}'_t([0,\overline{t}]))$.

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- The latter is an $\mathrm{SLE}_{\kappa'}$ and so the standard cut-point measure is well-defined on $\partial U \cap \partial V$.

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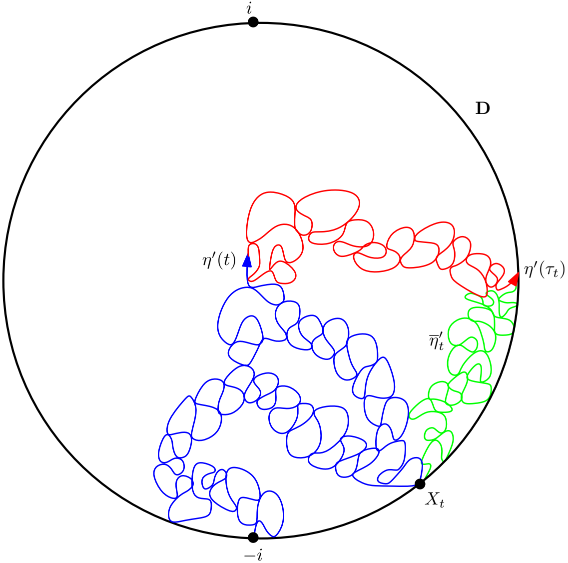

Construction of the cut-point measure

- To construct the $\mathrm{SLE}_{\kappa'}$ cut-point measure, we let $\mathcal{W} = (\mathbf{H}, h, 0, \infty)$ be a quantum wedge of weight $\frac{3\gamma^2}{2} - 2$ independent of the $\mathrm{SLE}_{\kappa'}$ η' in $\mathbf{H}$. For each Borel set $O \subseteq \mathbf{H}$ and $\epsilon > 0$, we let $N_\epsilon^{h,\eta'}(O)$ be the number of points in O which are closing points of connected components between the left and right outer boundaries of η' of quantum area at least ϵ.

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- Then, the sequence of measure $(\epsilon^{1-\kappa'/8} N_\epsilon^{h,\eta'})_{\epsilon>0}$ converges vaguely as $\epsilon \to 0$ almost surely. Denote the limit by $\mu_{h,\eta'}^{\text{cut}}$. If h^0 is a zero-boundary GFF on $\mathbf{H}$ independent of (h, η'), then the law of h^0 restricted to compact subsets of $\mathbf{H}$ is mutually absolutely continuous with respect to that of h and so the sequence of measures $(\epsilon^{1-\kappa'/8} N_\epsilon^{h^0,\eta'})_{\epsilon>0}$ converges vaguely as $\epsilon \to 0$ a.s. to some limit $\mu_{h^0,\eta'}^{\text{cut}}$.

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- The above convergence holds if we replace $\mathbf{H}$ by any simply connected domain $D \subseteq \mathbf{C}$ and η' is an $\mathrm{SLE}_{\kappa'}$ in D. Then, we set

$$\mu_{\eta'}^{\text{cut}}(dz) = r_D(z)\mathbf{E}[\mu_{h^0,\eta'}^{\text{cut}}(dz) \,|\, \eta'],$$

where $r_D(z) = \mathrm{CR}(z, D)^{2-8/\kappa'-\kappa'/8}$.

Thank you!