Quantitative ergodic theory

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Construction of quantitative OE: Følner tilings

Definition

Let Γ be an amenable group and (F_k) be a sequence of finite subsets of Γ. We call (F_k) a (left) **Følner tiling sequence** if the sequence of *tiles* (T_k) defined inductively by $T_0 = F_0$ and $T_{k+1} = T_k F_{k+1}$ satisfies the following conditions:

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1. (tiling condition) for all $k \in \mathbb{N}$, T_{k+1} is a *disjoint* union:

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Remark

The first condition amounts to saying that every element of T_k can uniquely be written as $f_0 \cdots f_k$ where each f_i belongs to F_i.

Profinite Følner tilings and profinite actions

Definition (Profinite Følner tilings)

A Følner tiling sequence $(F_k)_{k\in\mathbb{N}}$ is **profinite** if there exists a decreasing sequence of finite index subgroups Γ_k such that each F_k is a set of left coset representatives of Γ_{k-1} modulo Γ_k.

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Proposition

If (F_k) is a profinite Følner tiling sequence associated to (Γ_k), then the corresponding pmp action is isomorphic to the profinite action of Γ on $\varprojlim \Gamma/\Gamma_k$.

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Suppose that (F_k), (F'_k) *are* (ε_k, R_k), (ε'_k, R'_k) *Følner tiling sequences for* Γ *and* Γ', *such that* $|F_k| = |F'_k|$ *for all* $k \in \mathbb{N}$.

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Then the orbit equivalence coupling from Γ to Γ' is (φ, L^0)-integrable.

Applications

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.

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- *The lamplighter group and* $\mathbb{Z}$ *are* $(\log n)^{1-\varepsilon}$*-OE for all* $\varepsilon > 0$.

All OE are between profinite actions (Odometer-like).

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Which groups admit Følner tiling sequences ?

- Nilpotent groups: Yes (Delabie-Llosa-Tessera 24).

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- But: Polycyclic groups are (virtually) uniform lattices in connected Solvable Lie groups.
- connected Solvable Lie groups have probably always Følner tiling sequences.

Amandine Escalier's work

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

If Λ and Γ are (φ, L^0)-OE for some concave increasing function φ, then $Føl_\Lambda \circ \varphi \lesssim Føl_\Gamma$.

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For all increasing functions α and β find groups Λ and Γ such that

1. *$F\o l_\Lambda \approx \alpha$ and $F\o l_\Gamma \approx \alpha$;*

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For every convex increasing function β, there exists an (φ, L^0)-OE coupling from the group Γ_β to $\mathbb{Z}$, where φ is "nearly" β^{-1}: e.g. $(\beta^{-1})^{1-\varepsilon}$. The construction provides profinite actions (via Følner tilings).

Følner tilings for the lamplighter

- Lamplighter: $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} := \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$.
- standard generating set: $\{(0,1), (\delta_0, 0)\}$.
- The lamplighter point of view consists in viewing each element (f,n) of the group as a pair where f is a configuration of lamps, and where n is the position of the "lamplighter". Multiplying (f,n) on the right by the first generator amounts to moving the lamplighter from position n to $n+1$. Multiplying it by the second generator amounts to switching the light at position n.
- We define $F_0 = \{(f,n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \mathrm{supp}(f) \subseteq \{0,1\}, n \in \{0,1\}\}$ and

$$F_k = \left\{(f,0) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \mathrm{supp}(f) \subseteq [2^k, 2^{k+1}-1]\right\} \cup \left\{(f,2^k) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \mathrm{supp}(f) \subseteq [0, 2^k-1]\right\}.$$

- $T_k = \{(f,n) \in \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z} \colon \mathrm{supp}(f) \subseteq [0, 2^{k+1}-1], n \in [0, 2^{k+1}-1]\}$.

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Proposition

The group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ admits a (ε_k, R_k)-Følner tiling sequence $(F_k)_k$, with $|F_0| = 2^3$, and $|F_k| = 2 \cdot 2^{2^k}$, $R_k = 3 \cdot 2^{k+1}$ and $\varepsilon_k = 2^{-(k+1)}$ for $k \geq 1$.

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- To bound the diameter of T_k, observe that to join two elements (f,n) and (f',n') in T_n, the lamplighter may travel from position n to n', passing through the whole interval $[0, 2^{k+1}-1]$, while possibly switching all the lamps along the way.

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- If $s = (\delta_0, 0)$, then $T_k s = T_k$. If $s = (0,1)$, then

$$T_k s \setminus T_k = \{(f, 2^{k+1}) \in \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} \colon \operatorname{supp}(f) \subseteq [0, 2^{k+1}-1]\}.$$

So $|T_k s \setminus T_k| \leq 2^{2^{k+1}} = 2^{-(k+1)}|T_k|$, so we are done.

Corollary

The lamplighter group and $\mathbb{Z}$ are $(\log n)^{1-\varepsilon}$-OE for all $\varepsilon > 0$.

Beyond Følner tilings

Problem

Følner tilings provide OE which are at best almost L^1. Can we do better?

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

Baumslag-Solitar group: $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ *are* $(\exp, L^\infty)$*-OE.*

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- *Asymptotic dimension is not preserved under L^1-OE.*

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- An action of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ on $\prod_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$: $\mathbb{Z}$ acts by shift, $\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ acts coordinate-wise.
- An action of $\mathbb{Z}[1/2]$: for all $m \in \mathbb{Z}$, we decompose the space X as

$$X = \prod_{i<m} \mathbb{Z}/2\mathbb{Z} \times \prod_{i \geq m} \mathbb{Z}/2\mathbb{Z},$$

  and then $(2^m, 0)$ acts trivially on the first factor, and as the 2-adic odometer on the second factor.
- We extend it to an action of $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by shift.

Beyond Følner tilings: Sofic approximation

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- Example: Cayley graph $\mathcal{C}(\Gamma, S)$.
- Let $\mathcal{G}$ be a S-labeled graph. For $r \geq 1$, we denote

$$X^r = \left\{x \in X \mid B_{\mathcal{G}}(x, r) \simeq B_{\mathcal{C}(\Gamma,S)}(1_\Gamma, r)\right\},$$

  where $\simeq$ means isomorphic as S-labeled graphs.

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$$\lim_{n\to\infty} \mathbb{P}_{\mathcal{G}_n}\left(\mathcal{G}_n^{(r)}\right) = 1.$$

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- $\mathcal{G}_n$ is a Følner sequence.

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- Example: Cayley graph $\mathcal{C}(\Gamma, S)$.
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$$X^r = \left\{x \in X \mid B_{\mathcal{G}}(x, r) \simeq B_{\mathcal{C}(\Gamma,S)}(1_\Gamma, r)\right\},$$

where $\simeq$ means isomorphic as S-labeled graphs.

Definition (Sofic approximation)

Let $(\mathcal{G}_n)_n$ be a sequence of finite S-labeled graphs. $\mathbb{P}_{\mathcal{G}_n}$: renormalized counting measure on $\mathcal{G}_n$. $(\mathcal{G}_n)_n$ is a Sofic approximation of (Γ, S), if for every $r > 0$,

$$\lim_{n\to\infty} \mathbb{P}_{\mathcal{G}_n}\left(\mathcal{G}_n^{(r)}\right) = 1.$$

- $\mathcal{G}_n$ is a Følner sequence.
- $\mathcal{G}_n = \text{Schreier}(\Gamma/\Gamma_n, S)$, where Γ_n is a decreasing sequence of finite index subgroups such that $\bigcap_n \Gamma_n = \{1\}$.

Sofic approximation: construction of OE

- Let Γ and Λ be sofic groups, let $\mathcal{G}_n$ and $\mathcal{L}_n$ be sofic approximations.
- α, β decreasing functions such that $\lim_t \to \infty \alpha(t) = \lim_t \to \infty \beta(t) = 0$.

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Theorem (Carderi-Delabie-Koivisto-Le Maître-Tessera 23)

Let Γ and Λ be sofic groups.

- *if $F_n : \mathcal{G}_n \to \mathcal{L}_n$ is (α, β)-statistically bi-Lipschitz, then there exist pmp actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ and an OE between them.*

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- *if α and β satisfy*

$$\int \varphi(t)\alpha(t)dt < \infty \quad \text{and} \quad \int \psi(t)\beta(t)dt < \infty,$$

then the OE is (φ, ψ)-integrable.

Reminder on quantitative OE

Definition (Word distance on X)

Let Λ be a group generated by a finite subset S and let assume Λ acts freely on (X, μ), then the **word distance on** X associated to S is

$$d_S(x, x') = \min\{n \in \mathbb{N} \mid x' = s_1^{\pm 1} \dots s_n^{\pm 1} \cdot x\},$$

where $s_i \in S$ **if** x' **and** x **lie in a same orbit**,

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We use the measure μ to **compare the word distances** associated to two distinct pmp actions as follows:

Proposition (φ-integrable orbit equivalence)

Assume $\Lambda, \Gamma \curvearrowright (X,\mu)$ with same orbits. The actions are (φ,ψ)-OE iff for all $\lambda \in S_\Lambda$,

$$\int_X \varphi(d_{S_\Gamma}(x, \lambda \cdot x)) d\mu(x) < \infty,$$

and all $\gamma \in S_\Gamma$,

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Sketch of proof.

- Take a ultrafilter $\mathcal{U}$, and consider the limit $X = \lim_{\mathcal{U}} \mathcal{G}_n$ (similarly $Y = \lim_{\mathcal{U}} \mathcal{L}_n$).

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- The map $F_{\mathcal{U}} = \lim_{\mathcal{U}} F_n$ is a measure isomorphism and satisfies the conclusion of the lemma.
- In particular: $d_{S_\Gamma}(x, x') < \infty \iff d_{S_\Lambda}(F_{\mathcal{U}}(x), F_{\mathcal{U}}(x')) < \infty$: $F_{\mathcal{U}}$ is an OE.

Other constructions: wreath products

Definition

The wreath product of Λ with Γ:

$$\Lambda \wr \Gamma := \left(\bigoplus_{\Gamma} \Lambda \right) \rtimes \Gamma$$

(lamp group: Λ, base group: Γ)

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

If Γ_1 and Γ_2 admit a (φ, ψ)-integrable orbit equivalence coupling, and if Λ_1 and Λ_2 admit a (φ, ψ)-integrable orbit equivalence coupling, then the wreath products $\Lambda_1 \wr \Gamma_2$ and $\Lambda_2 \wr \Gamma_2$ also admit a (φ, ψ)-integrable orbit equivalence couplings.

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Corollary

Let $a, b \in \mathbb{N}$ with $a < b$ and let Δ be any finitely generated group, then there is an $(L^p, L^{p'})$-orbit equivalence coupling from $\Delta \wr \mathbb{Z}^b$ to $\Delta \wr \mathbb{Z}^a$ for every $p < \frac{a}{b}$ and $p' < \frac{b}{a}$.

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