Quantitative ergodic theory

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OE between free abelian groups

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

Let $d, k \in \mathbb{N}$. There is no L^p-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$ for $p > d/(d+k)$.

Today, we will focus on the following converse:

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Let $d, k' \in \mathbb{N}$. Then $\mathbb{Z}^d$ and $\mathbb{Z}^{d+k}$ are L^p-OE for all $p < d/(d+k)$.

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Problem

What happens for $p = d/(d+k)$?

Construction of quantitative OE: Følner tilings

Definition

Let Γ be an amenable group and (F_k) be a sequence of finite subsets of Γ. We call (F_k) a (left) **Følner tiling sequence** if the sequence of *tiles* (T_k) defined inductively by $T_0 = F_0$ and $T_{k+1} = T_k F_{k+1}$ satisfies the following conditions:

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Remark

The first condition amounts to saying that every element of T_k can uniquely be written as $f_0 \cdots f_k$ where each f_i belongs to F_i.

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- The writing $t = f_0 \cdots f_k$ (where $t \in T_k$), corresponds to the diadic decomposition of $t \in \mathbb{N}$:

$$t = \sum_{i=0}^{k} \varepsilon_i 2^i$$

where $\varepsilon_i = 0$ if $f_i = 0$, or $\varepsilon_i = 1$ if $f_i = 2^i$.

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Proposition

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Remark (Orbit equivalence)

Consider two amenable groups Γ and Γ' admitting respective Følner tiling sequences (F_k) and (F'_k) such that $|F_k| = |F'_k|$. Then the proposition provides us with free measure-preserving actions of Γ and Γ' on $X = \prod_k \{1, \ldots, |F_k|\}$, with same orbits.

Construction of the free pmp action

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- The proposition is proved!

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- As a measure space, $\mathbb{Z}_2 = \prod_k \{0, 2^k\}$.
- We recognize the pmp action of $\Gamma = \mathbb{Z}$ associated to $F_k = \{0, 2^k\}$, the tiles being $T_k = \{0, ..., 2^{k+1} - 1\}$, with the product measure.

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- This corresponds to $X = \mathbb{Z}_2 = \{\sum_{k=0}^{\infty} a_k 2^k \mid a_k \in \{0,1\}\}$, the ring of 2-adic numbers.
- The 2-odometer is the restriction to the subring $\mathbb{Z} \subset \mathbb{Z}_2$ of the action of $\mathbb{Z}_2$ on itself by addition.
- As a measure space, $\mathbb{Z}_2 = \prod_k \{0, 2^k\}$.
- We recognize the pmp action of $\Gamma = \mathbb{Z}$ associated to $F_k = \{0, 2^k\}$, the tiles being $T_k = \{0, ..., 2^{k+1} - 1\}$, with the product measure.
- Note that the presence of an initial sequence $(1,1,\ldots,1)$ (of length n) means that the corresponding $x_n = f_0 \ldots f_n$ lies in the boundary of T_n (hence adding 1 make it jump to the neighboring tile $2^n + T_n = [2^n, 2^{n+1} - 1]$.

Profinite Følner tilings and profinite actions

Definition (Følner tilings)

Let Γ be an amenable group and (F_k) be a sequence of finite subsets of Γ. We call (F_k) a (left) **Følner tiling sequence** if the sequence of *tiles* (T_k) defined inductively by by $T_0 = F_0$ and $T_{k+1} = T_k F_{k+1}$ satisfies the following conditions:

1. (tiling condition) for all $k \in \mathbb{N}$, T_{k+1} is a *disjoint* union:
$$T_{k+1} = \bigsqcup_{\gamma \in F_{k+1}} T_k \gamma;$$
2. (Fo lner condition) (T_k) is a left Følner sequence: for all $\gamma \in \Gamma$, $\lim_{k \to +\infty} \frac{|\gamma T_k \setminus T_k|}{|T_k|} = 0.$

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If in addition there exists a decreasing sequence of finite index subgroups Γ_k such that each F_k is a set of left coset representatives of Γ_{k-1} modulo Γ_k, then we call (F_k) a **profinite Følner tiling sequence** associated to (Γ_k).

Profinite Følner tilings and profinite actions

Definition (Profinite Følner tilings)

A Følner tiling sequence $(F_k)_{k\in\mathbb{N}}$ is **profinite** if there exists a decreasing sequence of finite index subgroups Γ_k such that each F_k is a set of left coset representatives of Γ_{k-1} modulo Γ_k.

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Proposition

If (F_k) is a profinite Følner tiling sequence associated to (Γ_k), then the corresponding pmp action is isomorphic to the profinite action of Γ on $\varprojlim \Gamma/\Gamma_k$.

Remark

We recover the fact that the 2-odometer is the action of $\mathbb{Z}$ on $\mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^k\mathbb{Z}$.

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Proof.

- The restriction of the projection $\Gamma \to \Gamma/\Gamma_n$ to T_n induces a bijection $\Phi_n : X_n \to \Gamma/\Gamma_n$.
- Since $F_n \subseteq \Gamma_{n-1}$, we have $\pi_{n-1}(g_n(x)) = \pi_{n-1}(g_{n-1}(x))$ for all $x \in X$.

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- Hence, the sequence (Φ_n) induces a map $\Phi : X \to \varprojlim \Gamma/\Gamma_k$, $x \mapsto (\pi_n(g_n(x)))$, which is an isomorphism of probability spaces.

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- By Equation (1), for a.e. $x \in X$ and all $\gamma \in \Gamma$ we have $g_n(\gamma \cdot x) = \gamma g_n(x)$ for all large enough n.

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- Hence Φ intertwines the two Γ-actions and we are done.

$\square$

Reminder on quantitative OE

Definition (Word distance on X)

Let Λ be a group generated by a finite subset S and let assume Λ acts freely on (X, μ), then the **word distance on** X associated to S is

$$d_S(x, x') = \min\{n \in \mathbb{N} \mid x' = s_1^{\pm 1} \dots s_n^{\pm 1} \cdot x\},$$

where $s_i \in S$ **if** x' **and** x **lie in a same orbit**,

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We use the measure μ to **compare the word distances** associated to two distinct pmp actions as follows:

Proposition (φ-integrable orbit equivalence)

Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ *with same orbits. The actions are* (φ, ψ)*-OE iff for all* $\lambda \in S_\Lambda$,

$$\int_X \varphi(d_{S_\Gamma}(x, \lambda \cdot x)) d\mu(x) < \infty,$$

and all $\gamma \in S_\Gamma$,

$$\int_X \psi(d_{S_\Lambda}(x, \gamma \cdot x)) d\mu(x) < \infty,$$

A distance for the cofinite equivalence relation

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$$\rho(x, x') = \inf\{n \in \mathbb{N} \colon \forall k \geq n, x_k = x'_k\}$$

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- Hence the cofinite equivalence relation is defined by $\rho(x, x') < +\infty$.
- we have that $\rho(\gamma \cdot x, x) > k$ if and only if $\gamma g_k(x) \notin T_k$.
- In particular,

$$\mu\left(\{x,\ \rho(\gamma \cdot x, x) > k\}\right) = \frac{|T_k \setminus \gamma^{-1} T_k|}{|T_k|} = \frac{|\gamma T_k \triangle T_k|}{2|T_k|}. \tag{3}$$

Quantitative Følner tiling sequences

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A Følner tiling sequence (F_k) of Γ is an (ε_k, R_k)-Følner tiling sequence if

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Let (F_k) be (ε_k, R_k)-Følner tiling sequence of a finitely generated group Γ equipped with a finite generating set S_Γ. Then

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Proof.

- The first item follows from Equation (3): $\mu(\{x,\ \rho(s \cdot x, x) > k\}) = \frac{|\gamma T_k \triangle T_k|}{2|T_k|}$.

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Proof.

- The first item follows from Equation (3): $\mu(\{x,\ \rho(s \cdot x, x) > k\}) = \frac{|\gamma T_k \Delta T_k|}{2|T_k|}$.
- For the second item, we simply observe that if $|\gamma|_{S_\Gamma} > 2R_k$, then $\gamma T_k \cap T_k = \emptyset$ as $\mathrm{diam}(T_k) \leq R_k$.

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Proposition

Suppose that (F_k), (F'_k) *are* (ε_k, R_k), (ε'_k, R'_k) *Følner tiling sequences for* Γ *and* Γ', *such that* $|F_k| = |F'_k|$ *for all* $k \in \mathbb{N}$.

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Then the orbit equivalence coupling from Γ to Γ' is (φ, L^0)-integrable.

- By the Lemma, for all $s \in S_\Gamma$ and for all $k \in \mathbb{N}$,

$$\mu\left(\{x \in X\colon d_{S_{\Gamma'}}(x, s\cdot x) > 2R'_k\}\right) \leq \mu\left(\{x \in X\colon \rho(s\cdot x, x) > k\}\right) \leq \varepsilon_k.$$

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- Using that φ is increasing, $\int_X \varphi(d_{S_{\Gamma'}}(x, s\cdot x))d\mu(x)$ is less than

$$\varphi(2R'_0) + \sum_{k=1}^{\infty} \varphi(2R'_k)\mu\left(\{x \in X\colon 2R'_{k-1} < d_{S_{\Gamma'}}(x, s\cdot x) \leq 2R'_k\}\right)$$

$$\leq \varphi(R'_0) + \sum_{k=1}^{\infty} \varphi(2R'_k)(\varepsilon_{k-1} - \varepsilon_k),$$

which is finite by assumption.

The case of abelian groups

Proposition

Let n be a positive integer. The group $\mathbb{Z}^n$ (equipped with its standard generating set) admits a profinite (ε_k, R_k)-Følner tiling sequence (F_k), with $|F_k| = 2^n$, $R_k = n2^{k+1}$ and $\varepsilon_k = 2^{-(k+1)}$ for any $k \geq 0$.

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- Finally take s a generator of $\mathbb{Z}^n$. Without loss of generality, we can assume that s is the first basis vector in $\mathbb{Z}^n$.
- Then, we have

$$T_k \setminus ((1, 0, \ldots, 0) + T_k) = \{0\} \times \{0, 1, \ldots, 2^{k+1} - 1\}^{n-1},$$

whose cardinality is 2^{k+1} smaller than that of T_k, so we are done. □

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Corollary

Let n and m be positive integers. The group $\mathbb{Z}^n$ admits a (ε_k, R_k)-Følner tiling sequence (F'_k), with $|F'_k| = 2^{nm}$, $R_k = n2^{m(k+1)}$ and $\varepsilon_k = 2^{-m(k+1)}$ for any $k \geq 0$.

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Proof of Corollary.

- Let $(F_k)_k$ be the Følner tilling sequence given in the proposition and for any $k \geq 0$ let $F'_k = F_{mk}F_{mk+1}\dots F_{mk+m-1}$. Note that $F'_k = \{0, 2^{mk}, 2\cdot 2^{mk}, \dots, (2^m-1)2^{mk}\}^n$ and $T'_k = \{0, 1, \dots, 2^{mk+m}-1\}^n$.

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- As T'_k is the set T_{mk+m-1} from the proposition, we have that the diameter of T'_k is at most $n2^{mk+m}$ and the set $T'_k \setminus (s + T'_k)$ has cardinality at most $2^{-mk-m}|T'_k|$ for any standard generator s of $\mathbb{Z}^n$.

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Theorem

There exists an OE from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ which is $(\varphi_\epsilon, \psi_\epsilon)$-integrable for every $\varepsilon > 0$, where

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The case of abelian groups

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