Quantitative ergodic theory

Romain Tessera

CNRS, Université Paris Cité et Sorbonne Université

20/02/24

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits.

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,

$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$

  is **integrable**,

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
  $$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
  is **integrable**,
- for all $\gamma \in \Gamma$,
  $$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
  is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for* $\varphi(t) = \psi(t) = t^p$*, this means in* L^p*-OE.*
- L^0*-OE: no integrability condition.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for* $\varphi(t) = \psi(t) = t^p$*, this means in* L^p*-OE.*
- L^0*-OE: no integrability condition.*
- *The* **faster** φ *tends to infinity, the* **stronger** *the condition is.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** φ *tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE) \Rightarrow (L^{1/2} - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
  $$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
  is **integrable**,
- for all $\gamma \in \Gamma$,
  $$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
  is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE) \Rightarrow (L^{1/2} - OE) \Rightarrow (\log - OE).$$

Følner profile and Orbit equivalence

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{ |A| \mid \frac{|As \bigtriangleup A|}{|A|} \leq 1/n, \ \forall s \in S \right\}$$

Følner profile and Orbit equivalence

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{ |A| \mid \frac{|As \triangle A|}{|A|} \leq 1/n,\ \forall s \in S \right\}$$

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $F\o l_\Lambda \approx F\o l_\Gamma$.*

Følner profile and Orbit equivalence

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min \left\{ |A| \mid \frac{|As \bigtriangleup A|}{|A|} \leq 1/n, \ \forall s \in S \right\}$$

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $F\o l_\Lambda \approx F\o l_\Gamma$.*
- *More generally, if Λ and Γ are (φ, L^0)-OE for some concave increasing function φ, then*

Følner profile and Orbit equivalence

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$Føl(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $Føl_\Lambda \approx Føl_\Gamma$.*
- *More generally, if Λ and Γ are (φ, L^0)-OE for some concave increasing function φ, then*

$$Føl_\Lambda \circ \varphi \lesssim Føl_\Gamma.$$

Følner profile and Orbit equivalence

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $F\o l_\Lambda \approx F\o l_\Gamma$.*
- *More generally, if Λ and Γ are (φ, L^0)-OE for some concave increasing function φ, then*

$$F\o l_\Lambda \circ \varphi \lesssim F\o l_\Gamma.$$

The second statement implies the first one: Λ and Γ are L^1-OE means that there exists a $(\mathrm{id}, \mathrm{id})$-OE from Λ to Γ. Note that id is concave and increasing (!).

Preparation for the proof (1): Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l_{\Lambda}(n) = \min\left\{|A| \mid \frac{|As \triangle A|}{|A|} \leq 1/n, \ \forall s \in S\right\}$$

Preparation for the proof (1): Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l_{\Lambda}(n) = \min\left\{|A| \mid \frac{|As \triangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Given a function $f : \Lambda \to \mathbb{R}$, we define the ℓ^1-norm of its (right) gradient by the equation

$$\|\nabla_S f\|_1 \quad := \quad \max_{s \in S} \sum_{g \in \Lambda} |f(gs) - f(g)|$$

Preparation for the proof (1): Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l_{\Lambda}(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Given a function $f : \Lambda \to \mathbb{R}$, we define the ℓ^1-norm of its (right) gradient by the equation

$$\begin{aligned} \|\nabla_S f\|_1 &:= \max_{s\in S} \sum_{g\in\Lambda} |f(gs) - f(g)| \\ &= \max_{s\in S} \|\rho(s)f - f\|_1, \end{aligned}$$

where ρ is the action by right translations of Λ on $\ell^1(\Lambda)$.

Preparation for the proof (1): Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l_{\Lambda}(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Given a function $f : \Lambda \to \mathbb{R}$, we define the ℓ^1-norm of its (right) gradient by the equation

$$\begin{aligned} \|\nabla_S f\|_1 &:= \max_{s\in S} \sum_{g\in\Lambda} |f(gs) - f(g)| \\ &= \max_{s\in S} \|\rho(s)f - f\|_1, \end{aligned}$$

where ρ is the action by right translations of Λ on $\ell^1(\Lambda)$.

Lemma (Folklore)

$$F\o l_{\Lambda}(n) = \min\left\{|\mathrm{supp}(f)| \mid \frac{\|\nabla_S f\|_1}{\|f\|_1} \leq 1/n\right\}.$$

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$.

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

  $$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

General strategy:

(1) start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$;

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

General strategy:

(1) start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$;

(2) extend it to a function $F : \Gamma \times X \to \mathbb{R}$;

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

General strategy:

(1) start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$;

(2) extend it to a function $F : \Gamma \times X \to \mathbb{R}$;

(3) use the action of Λ on $\Gamma \times X$ to define $\left\|\nabla_{S_\Lambda} F\right\|_1$.

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

  for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

General strategy:

(1) start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$;

(2) extend it to a function $F : \Gamma \times X \to \mathbb{R}$;

(3) use the action of Λ on $\Gamma \times X$ to define $\left\|\nabla_{S_\Lambda} F\right\|_1$.

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

Preparation for the proof (2): cocycle

- $\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:
$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$
for a.e. $x \in X$, $\lambda \in \Lambda$.
- "cocycle relation": $\alpha(\lambda'\lambda, x) = \alpha(\lambda', \lambda \cdot x)\alpha(\lambda, x)$.
- action of Λ on $\Gamma \times X$: $\lambda \star (\gamma, x) := (\alpha(\lambda, x)\gamma, \lambda \cdot x)$: this action is free, measure-preserving, with fundamental domain $\{1\} \times X$.

General strategy:

(1) start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$;

(2) extend it to a function $F : \Gamma \times X \to \mathbb{R}$;

(3) use the action of Λ on $\Gamma \times X$ to define $\left\|\nabla_{S_\Lambda} F\right\|_1$.

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\left\|\nabla_{S_\Lambda} F\right\|_1$.

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.
(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.
(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.
(3) define

$$\begin{aligned} \left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \|F(s \star \cdot) - F\|_1 \end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.
(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.
(3) define

$$\begin{aligned} \left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \|F(s \star \cdot) - F\|_1 \end{aligned}$$

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.

(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.

(3) define

$$\left\|\nabla_{S_\Lambda} F\right\|_1 = \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx$$
$$= \max_{s \in S_\Lambda} \|F(s \star \cdot) - F\|_1$$

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

$$\left\|\nabla_{S_\Lambda} F\right\|_1 = \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx$$

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.
(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.
(3) define

$$\begin{aligned} \|\nabla_{S_\Lambda} F\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \|F(s \star \cdot) - F\|_1 \end{aligned}$$

(4) use the integrability of the cocycle to dominate $\|\nabla_{S_\Lambda} F\|_1$ by $\|\nabla_{S_\Gamma} f\|_1$.

$$\begin{aligned} \|\nabla_{S_\Lambda} F\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(\alpha(s, x)\gamma, s \cdot x)) - F(\gamma, x)| dx \end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.
(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.
(3) define

$$\begin{aligned}\|\nabla_{S_\Lambda} F\|_1 &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(s\star(\gamma,x)) - F(\gamma,x)|dx \\ &= \max_{s\in S_\Lambda} \|F(s\star\cdot) - F\|_1\end{aligned}$$

(4) use the integrability of the cocycle to dominate $\|\nabla_{S_\Lambda} F\|_1$ by $\|\nabla_{S_\Gamma} f\|_1$.

$$\begin{aligned}\|\nabla_{S_\Lambda} F\|_1 &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(s\star(\gamma,x)) - F(\gamma,x)|dx \\ &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(\alpha(s,x)\gamma, s\cdot x)) - F(\gamma,x)|dx \\ &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |f(\gamma^{-1}\alpha(s,x)^{-1}) - f(\gamma^{-1})|dx\end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.

(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.

(3) define

$$\left\|\nabla_{S_\Lambda} F\right\|_1 = \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(s\star(\gamma,x)) - F(\gamma,x)|dx = \max_{s\in S_\Lambda} \|F(s\star\cdot) - F\|_1$$

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

$$\begin{aligned}\left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(s\star(\gamma,x)) - F(\gamma,x)|dx \\ &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |F(\alpha(s,x)\gamma, s\cdot x)) - F(\gamma,x)|dx \\ &= \max_{s\in S_\Lambda} \int_X \sum_\gamma |f(\gamma^{-1}\alpha(s,x)^{-1}) - f(\gamma^{-1})|dx \\ &= \max_{s\in S_\Lambda} \int_X \|\rho(\alpha(s,x)^{-1})f - f\|_1 dx\end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

(1) Start with a function $f : \Gamma \to \mathbb{R}$ realizing $F\o l_\Gamma(k)$.

(2) "Extend" it to a function $F : \Gamma \times X \to \mathbb{R}$ by $F(\gamma, x) = f(\gamma^{-1})$.

(3) define

$$\begin{aligned} \left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \|F(s \star \cdot) - F\|_1 \end{aligned}$$

(4) use the integrability of the cocycle to dominate $\left\|\nabla_{S_\Lambda} F\right\|_1$ by $\left\|\nabla_{S_\Gamma} f\right\|_1$.

$$\begin{aligned} \left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(s \star (\gamma, x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |F(\alpha(s, x)\gamma, s \cdot x)) - F(\gamma, x)| dx \\ &= \max_{s \in S_\Lambda} \int_X \sum_\gamma |f(\gamma^{-1}\alpha(s, x)^{-1}) - f(\gamma^{-1})| dx \\ &= \max_{s \in S_\Lambda} \int_X \|\rho(\alpha(s, x)^{-1}) f - f\|_1 dx \end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\|\rho(\alpha(s,x)^{-1})f - f\|_1 \quad = \quad \|\rho(t_1 \dots t_n)f - f\|_1$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\begin{aligned} \|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\ &\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\begin{aligned} \|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\ &\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \\ &= \sum_{i=1}^{n} \|\rho(t_i)f - f\|_1 \end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\begin{aligned}
\|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\
&\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \\
&= \sum_{i=1}^{n} \|\rho(t_i)f - f\|_1 \\
&\leq n \max_{t \in S_\Gamma} \|\rho(t)f - f\|_1
\end{aligned}$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$
\begin{aligned}
\|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\
&\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \\
&= \sum_{i=1}^{n} \|\rho(t_i)f - f\|_1 \\
&\leq n \max_{t \in S_\Gamma} \|\rho(t)f - f\|_1 \\
&= |\alpha(s, \lambda \cdot x)|_{S_\Gamma} \left\|\nabla_{S_\Gamma} f\right\|_1
\end{aligned}
$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\begin{aligned}
\|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\
&\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \\
&= \sum_{i=1}^{n} \|\rho(t_i)f - f\|_1 \\
&\leq n \max_{t \in S_\Gamma} \|\rho(t)f - f\|_1 \\
&= |\alpha(s, \lambda \cdot x)|_{S_\Gamma} \left\|\nabla_{S_\Gamma} f\right\|_1
\end{aligned}$$

Inject it:

$$\left\|\nabla_{S_\Lambda} F\right\|_1 = \max_{s \in S_\Lambda} \int_X \|\rho(\alpha(s,x)^{-1})f - f\|_1 dx$$

Proof of the Theorem: case of $\varphi(t) = t$

For each x, write $\alpha(s, \lambda \cdot x)^{-1} = t_n \dots t_1$, where $t_i \in S_\Gamma$, and $n = |\alpha(s, \lambda \cdot x)|_{S_\Gamma}$.

$$\begin{aligned}
\|\rho(\alpha(s,x)^{-1})f - f\|_1 &= \|\rho(t_1 \dots t_n)f - f\|_1 \\
&\leq \sum_{i=1}^{n} \|\rho(t_i \dots t_1)f - \rho(t_{i-1} \dots t_1)f\|_1 \\
&= \sum_{i=1}^{n} \|\rho(t_i)f - f\|_1 \\
&\leq n \max_{t \in S_\Gamma} \|\rho(t)f - f\|_1 \\
&= |\alpha(s, \lambda \cdot x)|_{S_\Gamma} \left\|\nabla_{S_\Gamma} f\right\|_1
\end{aligned}$$

Inject it:

$$\begin{aligned}
\left\|\nabla_{S_\Lambda} F\right\|_1 &= \max_{s \in S_\Lambda} \int_X \|\rho(\alpha(s,x)^{-1})f - f\|_1 dx \\
&\leq \left(\max_{s \in S_\Lambda} \int_X |\alpha(s, \lambda \cdot x)|_{S_\Gamma}\right) \left\|\nabla_{S_\Gamma} f\right\|_1
\end{aligned}$$

Let $C := max_{s \in S_\Lambda} \int_X |\alpha(s, \lambda \cdot x)|_{S_\Gamma} < \infty$, we get: $\left\|\nabla_{S_\Lambda} F\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\left\|\nabla_{S_\Lambda} F\right\|_1$.

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\|\nabla_{S_\Lambda} F\|_1$.
Recall that $\{1_\Gamma\} \times X$ is a fundamental domain for the action of Λ. So write for each $x \in X$, $F(\lambda \star (1_\Gamma, x)) = f^x(\lambda)$. Note that

$$\|\nabla_{S_\Lambda} F\|_1 = \int_X \|\nabla_{S_\Lambda} f^x\|_1 \, dx$$

Hence there exists x such that $\|\nabla_{S_\Lambda} f^x\|_1 \leq \|\nabla_{S_\Lambda} F\|_1$

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\|\nabla_{S_\Lambda} F\|_1$.
Recall that $\{1_\Gamma\} \times X$ is a fundamental domain for the action of Λ. So write for each $x \in X$, $F(\lambda \star (1_\Gamma, x)) = f^x(\lambda)$. Note that

$$\|\nabla_{S_\Lambda} F\|_1 = \int_X \|\nabla_{S_\Lambda} f^x\|_1 \, dx$$

Hence there exists x such that $\|\nabla_{S_\Lambda} f^x\|_1 \leq \|\nabla_{S_\Lambda} F\|_1 \leq C \|\nabla_{S_\Gamma} f\|_1$.

(6) Now we must show that $\|f_x\| \geq \|f\|$, and that $|\mathrm{supp}(f_x)| \leq |\mathrm{supp}(f)|$.

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\left\|\nabla_{S_\Lambda} F\right\|_1$.
Recall that $\{1_\Gamma\} \times X$ is a fundamental domain for the action of Λ. So write for each $x \in X$, $F(\lambda \star (1_\Gamma, x)) = f^x(\lambda)$. Note that

$$\left\|\nabla_{S_\Lambda} F\right\|_1 = \int_X \left\|\nabla_{S_\Lambda} f^x\right\|_1 dx$$

Hence there exists x such that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq \left\|\nabla_{S_\Lambda} F\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.

(6) Now we must show that $\|f_x\| \geq \|f\|$, and that $|\mathrm{supp}(f_x)| \leq |\mathrm{supp}(f)|$. But we actually have equality in both cases!

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\|\nabla_{S_\Lambda} F\|_1$.
Recall that $\{1_\Gamma\} \times X$ is a fundamental domain for the action of Λ. So write for each $x \in X$, $F(\lambda \star (1_\Gamma, x)) = f^x(\lambda)$. Note that

$$\|\nabla_{S_\Lambda} F\|_1 = \int_X \|\nabla_{S_\Lambda} f^x\|_1 \, dx$$

Hence there exists x such that $\|\nabla_{S_\Lambda} f^x\|_1 \leq \|\nabla_{S_\Lambda} F\|_1 \leq C \|\nabla_{S_\Gamma} f\|_1$.

(6) Now we must show that $\|f_x\| \geq \|f\|$, and that $|\mathrm{supp}(f_x)| \leq |\mathrm{supp}(f)|$. But we actually have equality in both cases! Indeed:

$$f^x(\lambda) = F(\lambda \star (1_\Gamma, x)) = F(\alpha(\lambda, x), x) = f(\alpha(x, \lambda)^{-1}).$$

Proof of the Theorem: case of $\varphi(t) = t$

(5) by pigeonhole, find a Λ-orbit such that the restriction of F defines a function on Λ whose gradient is controlled by $\|\nabla_{S_\Lambda} F\|_1$.
Recall that $\{1_\Gamma\} \times X$ is a fundamental domain for the action of Λ. So write for each $x \in X$, $F(\lambda \star (1_\Gamma, x)) = f^x(\lambda)$. Note that

$$\|\nabla_{S_\Lambda} F\|_1 = \int_X \|\nabla_{S_\Lambda} f^x\|_1 \, dx$$

Hence there exists x such that $\|\nabla_{S_\Lambda} f^x\|_1 \leq \|\nabla_{S_\Lambda} F\|_1 \leq C \|\nabla_{S_\Gamma} f\|_1$.

(6) Now we must show that $\|f_x\| \geq \|f\|$, and that $|\mathrm{supp}(f_x)| \leq |\mathrm{supp}(f)|$. But we actually have equality in both cases! Indeed:

$$f^x(\lambda) = F(\lambda \star (1_\Gamma, x)) = F(\alpha(\lambda, x), x) = f(\alpha(x, \lambda)^{-1}).$$

Since $\alpha(\cdot, x)$ is a bijection, we deduce that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

– Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

– Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.

– We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

– Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.

– We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.

– We have proved (6) that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

– Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.

– We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.

– We have proved (6) that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.

– Hence:

$$F\o l_\Gamma(k) = |\mathrm{supp}(f)|$$

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

- Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.
- We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.
- We have proved (6) that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.
- Hence:

$$F\o l_\Gamma(k) = |\mathrm{supp}(f)| = |\mathrm{supp}(f^x)|$$

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

- Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.
- We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.
- We have proved (6) that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.
- Hence:

$$F\o l_\Gamma(k) = |\mathrm{supp}(f)| = |\mathrm{supp}(f^x)| \geq F\o l_\Lambda\left(\frac{\|f^x\|}{\left\|\nabla_{S_\Lambda} f^x\right\|_1}\right)$$

Proof of the Theorem: case of $\varphi(t) = t$

(7) Put things together:

- Since f realizes $F\o l_\Gamma(k)$, we have $k = \frac{\|f\|}{\left\|\nabla_{S_\Gamma} f\right\|_1}$ and $F\o l_\Gamma(k) = |\mathrm{supp}(f)|$.
- We have proved (5) that $\left\|\nabla_{S_\Lambda} f^x\right\|_1 \leq C \left\|\nabla_{S_\Gamma} f\right\|_1$.
- We have proved (6) that $\|f_x\| = \|f\|$ and $|\mathrm{supp}(f_x)| = |\mathrm{supp}(f)|$.
- Hence:

$$F\o l_\Gamma(k) = |\mathrm{supp}(f)| = |\mathrm{supp}(f^x)| \geq F\o l_\Lambda\left(\frac{\|f^x\|}{\left\|\nabla_{S_\Lambda} f^x\right\|_1}\right) \geq F\o l_\Lambda(k/C).$$

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*

Optimality of the theorem

Corollary

- *Let* $d, k \in \mathbb{N}$. *If there exists an* (L^p, L^0)*-OE from* $\mathbb{Z}^{d+k}$ *to* $\mathbb{Z}^d$, *then* $p \leq d/(d+k)$.
- *Let* $d, k \in \mathbb{N}$. *If there exists an* (L^p, L^0)*-OE from* $F \wr \mathbb{Z}^{d+k}$ *to* $F \wr \mathbb{Z}^d$, *then* $p \leq d/(d+k)$.

Optimality of the theorem

Corollary

- *Let* $d, k \in \mathbb{N}$. *If there exists an* (L^p, L^0)*-OE from* $\mathbb{Z}^{d+k}$ *to* $\mathbb{Z}^d$, *then* $p \leq d/(d+k)$.
- *Let* $d, k \in \mathbb{N}$. *If there exists an* (L^p, L^0)*-OE from* $F \wr \mathbb{Z}^{d+k}$ *to* $F \wr \mathbb{Z}^d$, *then* $p \leq d/(d+k)$.
- *If* Λ *has exponential growth and if there is a* (φ, L^0)*-OE from* Λ *to* $\mathbb{Z}$, *then* $\varphi(n) \lesssim \log n$.

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

First case: applying the theorem with $\varphi(t) = t^p$, we have $F\o l_{\mathbb{Z}^{d+k}}(t^p) \lesssim F\o l_{\mathbb{Z}^d}(t)$. Recall that $F\o l_{\mathbb{Z}^d}(t) \approx t^d$. Hence we get $t^{p(d+k)} \lesssim t^d$, from which we deduce that $p(d+k) \leq d$, and $p \leq d/(d+k)$.

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

First case: applying the theorem with $\varphi(t) = t^p$, we have $F\o l_{\mathbb{Z}^{d+k}}(t^p) \lesssim F\o l_{\mathbb{Z}^d}(t)$. Recall that $F\o l_{\mathbb{Z}^d}(t) \approx t^d$. Hence we get $t^{p(d+k)} \lesssim t^d$, from which we deduce that $p(d+k) \leq d$, and $p \leq d/(d+k)$.

Second case: same using that $F\o l_{F \wr \mathbb{Z}^d}(t) \approx e^{t^d}$.

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

First case: applying the theorem with $\varphi(t) = t^p$, we have $F\o l_{\mathbb{Z}^{d+k}}(t^p) \lesssim F\o l_{\mathbb{Z}^d}(t)$. Recall that $F\o l_{\mathbb{Z}^d}(t) \approx t^d$. Hence we get $t^{p(d+k)} \lesssim t^d$, from which we deduce that $p(d+k) \leq d$, and $p \leq d/(d+k)$.

Second case: same using that $F\o l_{F\wr\mathbb{Z}^d}(t) \approx e^{t^d}$.

Third case: using $Vol_\Lambda \lesssim F\o l_\Lambda$, we get by the theorem $\exp \circ \varphi(t) \lesssim t$. So $\varphi(t) \lesssim \log t$.

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let $d, k' \in \mathbb{N}$. Then $\mathbb{Z}^d$ and $\mathbb{Z}^{d+k}$ are L^p-OE for all $p < d/(d+k)$.*

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let $d, k' \in \mathbb{N}$. Then $\mathbb{Z}^d$ and $\mathbb{Z}^{d+k}$ are L^p-OE for all $p < d/(d+k)$.*
- *Let $d, k' \in \mathbb{N}$. Then $F \wr \mathbb{Z}^{d+k}$ and $F \wr \mathbb{Z}^d$ are L^p-OE for all $p < d/(d+k)$.*

Optimality of the theorem

Corollary

- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.*
- *If Λ has exponential growth and if there is a (φ, L^0)-OE from Λ to $\mathbb{Z}$, then $\varphi(n) \lesssim \log n$.*

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let $d, k' \in \mathbb{N}$. Then $\mathbb{Z}^d$ and $\mathbb{Z}^{d+k}$ are L^p-OE for all $p < d/(d+k)$.*
- *Let $d, k' \in \mathbb{N}$. Then $F \wr \mathbb{Z}^{d+k}$ and $F \wr \mathbb{Z}^d$ are L^p-OE for all $p < d/(d+k)$.*
- *The lamplighter group and $\mathbb{Z}$ are $\log n^{1-\varepsilon}$-OE for all $\varepsilon > 0$.*

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as: $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2**-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a\cdot(0,0,0,1,\ldots)=(1,0,0,1\ldots)$
  $a\cdot(1,0,0,\ldots)=(0,1,0,\ldots)$
  $a\cdot(1,1,1,0,\ldots)=(0,0,0,1,\ldots)$
- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a\cdot(1,2,0,3,\ldots)=(2,2,0,3,\ldots)$
  $a\cdot(3,1,2,0,\ldots)=$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2**-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Two sequences belong to the **same orbit** if and only if they differ by at most finitely many coordinates.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a\cdot(0,0,0,1,\ldots)=(1,0,0,1\ldots)$
  $a\cdot(1,0,0,\ldots)=(0,1,0,\ldots)$
  $a\cdot(1,1,1,0,\ldots)=(0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a\cdot(1,2,0,3,\ldots)=(2,2,0,3,\ldots)$
  $a\cdot(3,1,2,0,\ldots)=(0,2,2,0,\ldots)$
  $a\cdot(3,3,3,3,1,0,\ldots)=(0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Two sequences belong to the **same orbit** if and only if they differ by at most finitely many coordinates.

The d-odometer is the action of $\mathbb{Z}$ by translation on $\mathbb{Z}_d$ (the ring of d-adic numbers).

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

Example: if $x = (0,1,1,\ldots)$, $y = (1,0,1,\ldots)$, then

$$F(x,y) = (0+2, 1+0, 1+2, \ldots) = (2,1,3,\ldots).$$

Proposition

This OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ is $(L^{1/2-\varepsilon}, L^{2-\varepsilon})$ for all $\varepsilon > 0$.

A few open problems

- Prove that the bounds given by the Corollary are optimal.

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.
- Find a OE between different odometers over $\mathbb{Z}$ (or $\mathbb{Z}^d$) with good integrability conditions.

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.
- Find a OE between different odometers over $\mathbb{Z}$ (or $\mathbb{Z}^d$) with good integrability conditions. E.g. construct an OE from the 2-odometer to the 3-odometer.

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.
- Find a OE between different odometers over $\mathbb{Z}$ (or $\mathbb{Z}^d$) with good integrability conditions. E.g. construct an OE from the 2-odometer to the 3-odometer.
- What is the best integrability of an OE between the Bernoulli shifts $\{1,2\}^{\mathbb{Z}}$ and $\{1,2,3\}^{\mathbb{Z}}$?

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.
- Find a OE between different odometers over $\mathbb{Z}$ (or $\mathbb{Z}^d$) with good integrability conditions. E.g. construct an OE from the 2-odometer to the 3-odometer.
- What is the best integrability of an OE between the Bernoulli shifts $\{1, 2\}^{\mathbb{Z}}$ and $\{1, 2, 3\}^{\mathbb{Z}}$? Partial answer: cannot be log-integrable (Kerr-Li), because the entropy is preserved.

A few open problems

- Prove that the bounds given by the Corollary are optimal. E.g. construct an OE from $\mathbb{Z}^2$ to $\mathbb{Z}$ which is $(L^{1/2}, L^0)$.
- Find a OE between different odometers over $\mathbb{Z}$ (or $\mathbb{Z}^d$) with good integrability conditions. E.g. construct an OE from the 2-odometer to the 3-odometer.
- What is the best integrability of an OE between the Bernoulli shifts $\{1,2\}^{\mathbb{Z}}$ and $\{1,2,3\}^{\mathbb{Z}}$? Partial answer: cannot be log-integrable (Kerr-Li), because the entropy is preserved.
- What is the best integrability of an OE between the Bernoulli shifts $\{1,2\}^{\mathbb{Z}}$ and $\{1,2\}^{\mathbb{Z}^2}$?