Quantitative ergodic theory

Romain Tessera

CNRS, Université Paris Cité et Sorbonne Université

20/02/24

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),
- (X, μ) **probability space** (example: (S^1, λ) where λ is Lebesgue measure, $\{0,1\}^\Lambda$, equipped with the product measure),

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),
- (X, μ) **probability space** (example: (S^1, λ) where λ is Lebesgue measure, $\{0,1\}^\Lambda$, equipped with the product measure),
- An **pmp action** $\Lambda \curvearrowright (X, \mu)$: i.e. a free measure-preserving action.

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),
- (X, μ) **probability space** (example: (S^1, λ) where λ is Lebesgue measure, $\{0,1\}^\Lambda$, equipped with the product measure),
- An **pmp action** $\Lambda \curvearrowright (X, \mu)$: i.e. a free measure-preserving action.

Exemples

- *Rotations*: $\mathbb{Z} \curvearrowright (S^1, \lambda)$ generated by an irrational rotation,

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),
- (X, μ) **probability space** (example: (S^1, λ) where λ is Lebesgue measure, $\{0,1\}^\Lambda$, equipped with the product measure),
- An **pmp action** $\Lambda \curvearrowright (X, \mu)$: i.e. a free measure-preserving action.

Exemples

- *Rotations*: $\mathbb{Z} \curvearrowright (S^1, \lambda)$ generated by an irrational rotation,
- *Bernoulli shift*: $\Lambda \curvearrowright \{0,1\}^\Lambda$.

Group actions preserving a probability

- Λ a **countable group** (examples: $\mathbb{Z}$, $\mathbb{Z}^d$, free group on k generators F_k),
- (X, μ) **probability space** (example: (S^1, λ) where λ is Lebesgue measure, $\{0,1\}^\Lambda$, equipped with the product measure),
- An **pmp action** $\Lambda \curvearrowright (X, \mu)$: i.e. a free measure-preserving action.

Exemples

- *Rotations*: $\mathbb{Z} \curvearrowright (S^1, \lambda)$ generated by an irrational rotation,
- *Bernoulli shift*: $\Lambda \curvearrowright \{0,1\}^\Lambda$.

Context:

- Ergodic theory,
- Representation theory,
- Operator algebras,
- Percolation theory (probabilities),
- Lattices in Lie groups...

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X, \mu) \to (Y, \nu)$, and $\theta : \Lambda \to \Gamma$ such that

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X, \mu) \to (Y, \nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X, \mu) \to (Y, \nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Definition (Orbit equivalence)

Two (free) pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **orbit equivalent** (OE), if there exists an isomorphism (of measure spaces) $\Psi : (X,\mu) \to (Y,\nu)$ such that for a.e. $x \in X$,

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Definition (Orbit equivalence)

Two (free) pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **orbit equivalent** (OE), if there exists an isomorphism (of measure spaces) $\Psi : (X,\mu) \to (Y,\nu)$ such that for a.e. $x \in X$,

$$\Psi(\Lambda \cdot x) = \Gamma \cdot \Psi(x).$$

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Definition (Orbit equivalence)

Two (free) pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **orbit equivalent** (OE), if there exists an isomorphism (of measure spaces) $\Psi : (X,\mu) \to (Y,\nu)$ such that for a.e. $x \in X$,

$$\Psi(\Lambda \cdot x) = \Gamma \cdot \Psi(x).$$

- $\mathbb{Z} \curvearrowright (S^1, \lambda)$ and $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ are *not* isomorphic (spectrum);

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Definition (Orbit equivalence)

Two (free) pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **orbit equivalent** (OE), if there exists an isomorphism (of measure spaces) $\Psi : (X,\mu) \to (Y,\nu)$ such that for a.e. $x \in X$,

$$\Psi(\Lambda \cdot x) = \Gamma \cdot \Psi(x).$$

- $\mathbb{Z} \curvearrowright (S^1, \lambda)$ and $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ are *not* isomorphic (spectrum);
- $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ and $\mathbb{Z} \curvearrowright \{0,1,2\}^{\mathbb{Z}}$ are *not* isomorphic (Kolmogorov-Sinai).

Orbit equivalence

Definition (Isomorphism)

Two pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **isomorphic**, if there exist isomorphisms $\Psi : (X,\mu) \to (Y,\nu)$, and $\theta : \Lambda \to \Gamma$ such that for a.e. $x \in X$, and all $\lambda \in \Lambda$,

$$\Psi(\lambda \cdot x) = \theta(\lambda) \cdot \Psi(x).$$

Definition (Orbit equivalence)

Two (free) pmp actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are **orbit equivalent** (OE), if there exists an isomorphism (of measure spaces) $\Psi : (X,\mu) \to (Y,\nu)$ such that for a.e. $x \in X$,

$$\Psi(\Lambda \cdot x) = \Gamma \cdot \Psi(x).$$

- $\mathbb{Z} \curvearrowright (S^1, \lambda)$ and $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ are *not* isomorphic (spectrum);
- $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ and $\mathbb{Z} \curvearrowright \{0,1,2\}^{\mathbb{Z}}$ are *not* isomorphic (Kolmogorov-Sinai).
- Any two ergodic pmp actions of $\mathbb{Z}$ are OE (Dye 59).

Amenable groups

Definition

A countable group Λ is **amenable** if it admits a sequence of "almost-invariant finite subsets" $A_n \subset \Lambda$, i.e. such that for all $\lambda \in \Lambda$,

$$\frac{|A_n\lambda \vartriangle A_n|}{|A_n|} \to 0.$$

((A_n) is called a right Følner sequence)

Amenable groups

Definition

A countable group Λ is **amenable** if it admits a sequence of "almost-invariant finite subsets" $A_n \subset \Lambda$, i.e. such that for all $\lambda \in \Lambda$,

$$\frac{|A_n\lambda \bigtriangleup A_n|}{|A_n|} \to 0.$$

((A_n) is called a right Følner sequence)

Exemples

- $\mathbb{Z}^d$, with $A_n = [-n, n]^d$;

Amenable groups

Definition

A countable group Λ is **amenable** if it admits a sequence of "almost-invariant finite subsets" $A_n \subset \Lambda$, i.e. such that for all $\lambda \in \Lambda$,

$$\frac{|A_n\lambda \triangle A_n|}{|A_n|} \to 0.$$

((A_n) is called a right Følner sequence)

Exemples

- $\mathbb{Z}^d$, with $A_n = [-n, n]^d$;
- stable under extension, subgroup, quotient...

Amenable groups

Definition

A countable group Λ is **amenable** if it admits a sequence of "almost-invariant finite subsets" $A_n \subset \Lambda$, i.e. such that for all $\lambda \in \Lambda$,

$$\frac{|A_n\lambda \vartriangle A_n|}{|A_n|} \to 0.$$

((A_n) is called a right Følner sequence)

Exemples

- $\mathbb{Z}^d$, with $A_n = [-n, n]^d$;
- stable under extension, subgroup, quotient...
- free groups F_k on $k \geq 2$ generators **are not** amenable.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

Things are very different for **non-amenable** groups.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

Things are very different for **non-amenable** groups. For instance

Theorem (Gaboriau 00)

If F_k and $F_{k'}$ have OE pmp actions, then $k = k'$.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

Things are very different for **non-amenable** groups. For instance

Theorem (Gaboriau 00)

If F_k and $F_{k'}$ have OE pmp actions, then $k = k'$.

Problem

Is-this the end of the story for amenable groups?

To try to answer (negatively) this question, we address the following points:

- **quantify orbit equivalence**: add "constraints" on the orbit-equivalence relation.

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are OE.

Things are very different for **non-amenable** groups. For instance

Theorem (Gaboriau 00)

If F_k and $F_{k'}$ have OE pmp actions, then $k = k'$.

Problem

Is-this the end of the story for amenable groups?

To try to answer (negatively) this question, we address the following points:

- **quantify orbit equivalence**: add "constraints" on the orbit-equivalence relation.
- **quantify amenability**: e.g.: find a ways to say that $\mathbb{Z}$ is "more amenable" than $\mathbb{Z}^2$;

A famous theorem of Ornstein-Weiss

Theorem (Ornstein-Weiss 80)

Let Λ and Γ be two (infinite) countable amenable groups. Then any pmp ergodic actions $\Lambda \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ are OE.

Things are very different for **non-amenable** groups. For instance

Theorem (Gaboriau 00)

If F_k and $F_{k'}$ have OE pmp actions, then $k = k'$.

Problem

Is-this the end of the story for amenable groups?

To try to answer (negatively) this question, we address the following points:

- **quantify orbit equivalence**: add "constraints" on the orbit-equivalence relation.
- **quantify amenability**: e.g.: find a ways to say that $\mathbb{Z}$ is "more amenable" than $\mathbb{Z}^2$;

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$.

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$. (Similarly: $\beta : \Gamma \times X \to \Lambda$)

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$. (Similarly: $\beta : \Gamma \times X \to \Lambda$)

Hence for every x, $\alpha(\cdot, x)$ is a **bijection** between Λ to Γ.

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$. (Similarly: $\beta : \Gamma \times X \to \Lambda$)

Hence for every x, $\alpha(\cdot, x)$ is a **bijection** between Λ to Γ.

Note that $\beta(\alpha(x, \cdot), x) = \text{id}$.

Orbit equivalence cocycles

Problem: find a substitute for the lack of isomorphism between Λ and Γ.

Definition (Cocycle)

$\Lambda, \Gamma \curvearrowright X$ with (a.e.) same orbits. Define $\alpha : \Lambda \times X \to \Gamma$ by:

$$\alpha(\lambda, x) \cdot x = \lambda \cdot x,$$

for a.e. $x \in X$, $\lambda \in \Lambda$. (Similarly: $\beta : \Gamma \times X \to \Lambda$)

Hence for every x, $\alpha(\cdot, x)$ is a **bijection** between Λ to Γ.

Note that $\beta(\alpha(x, \cdot), x) = \mathrm{id}$.

Problem: Given generating sets S_Λ and S_Γ, quantify the "average distortion" of the "random" map $\alpha(\cdot, x) : \Lambda \to \Gamma$.

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits.

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,

  $$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$

  is **integrable**,

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is.*

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)-**OE** if

- for all $\lambda \in \Lambda$,
  $$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
  is **integrable**,
- for all $\gamma \in \Gamma$,
  $$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
  is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
$$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
is **integrable**,
- for all $\gamma \in \Gamma$,
$$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE) \Rightarrow (L^{1/2} - OE)$$

Quantifying orbit equivalence

Definition (φ orbit equivalence)

Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions tending to ∞. Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ with same orbits. **The actions are** (φ, ψ)**-OE** if

- for all $\lambda \in \Lambda$,
  $$x \mapsto \varphi(|\alpha(x, \lambda)|_{S_\Gamma})$$
  is **integrable**,
- for all $\gamma \in \Gamma$,
  $$x \mapsto \psi(|\beta(x, \gamma)|_{S_\Lambda})$$
  is **integrable**.

Remark

- *Note that for $\varphi(t) = \psi(t) = t^p$, this means in L^p-OE.*
- *L^0-OE: no integrability condition.*
- *The* **faster** *φ tends to infinity, the* **stronger** *the condition is. For instance:*

$$(L^\infty - OE) \Rightarrow (L^2 - OE) \Rightarrow (L^1 - OE) \Rightarrow (L^{1/2} - OE) \Rightarrow (\log - OE).$$

Quantifying orbit equivalence: other point of view

Definition (Word distance on X)

Let Λ be a group generated by a finite subset S and let assume Λ acts freely on (X, μ), then the **word distance on** X associated to S is

$$d_S(x, x') = \min\{n \in \mathbb{N} \mid x' = s_1^{\pm 1} \dots s_n^{\pm 1} \cdot x\},$$

where $s_i \in S$ **if** x' **and** x **lie in a same orbit**,

Quantifying orbit equivalence: other point of view

Definition (Word distance on X)

Let Λ be a group generated by a finite subset S and let assume Λ acts freely on (X, μ), then the **word distance on** X associated to S is

$$d_S(x, x') = \min\{n \in \mathbb{N} \mid x' = s_1^{\pm 1} \dots s_n^{\pm 1} \cdot x\},$$

where $s_i \in S$ **if** x' **and** x **lie in a same orbit**, and $d_S(x, x') = \infty$ otherwise.

Quantifying orbit equivalence: other point of view

Definition (Word distance on X)

Let Λ be a group generated by a finite subset S and let assume Λ acts freely on (X, μ), then the **word distance on** X associated to S is

$$d_S(x, x') = \min\{n \in \mathbb{N} \mid x' = s_1^{\pm 1} \dots s_n^{\pm 1} \cdot x\},$$

where $s_i \in S$ **if** x' **and** x **lie in a same orbit**, and $d_S(x, x') = \infty$ otherwise.

We use the measure μ to **compare the word distances** associated to two distinct pmp actions as follows:

Proposition (φ-integrable orbit equivalence)

Assume $\Lambda, \Gamma \curvearrowright (X, \mu)$ *with same orbits. The actions are* (φ, ψ)*-OE iff for all* $\lambda \in S_\Lambda$*,*

$$\int_X \varphi(d_{S_\Gamma}(x, \lambda \cdot x)) d\mu(x) < \infty,$$

and all $\gamma \in S_\Gamma$*,*

$$\int_X \psi(d_{S_\Lambda}(x, \gamma \cdot x)) d\mu(x) < \infty,$$

Growth function

Let Λ be a group generated by a finite subset S. Define the growth function of Λ

$$V_\Lambda(n) = |S^n| = \left\{ g \in \Lambda \mid g = s_1^{\pm 1} \dots s_n^{\pm 1},\ s_i \in S \right\}.$$

Growth function

Let Λ be a group generated by a finite subset S. Define the growth function of Λ

$$V_\Lambda(n) = |S^n| = \left\{ g \in \Lambda \mid g = s_1^{\pm 1} \dots s_n^{\pm 1},\ s_i \in S \right\}.$$

The asymptotic behavior of V_G does not depend on S.

Growth function

Let Λ be a group generated by a finite subset S. Define the growth function of Λ

$$V_\Lambda(n) = |S^n| = \left\{ g \in \Lambda \mid g = s_1^{\pm 1} \dots s_n^{\pm 1},\ s_i \in S \right\}.$$

The asymptotic behavior of V_G does not depend on S.

Exemples

- $V_{\mathbb{Z}^d}(n) \approx n^d$.

Growth function

Let Λ be a group generated by a finite subset S. Define the growth function of Λ

$$V_\Lambda(n) = |S^n| = \left\{ g \in \Lambda \mid g = s_1^{\pm 1} \dots s_n^{\pm 1},\ s_i \in S \right\}.$$

The asymptotic behavior of V_G does not depend on S.

Exemples

- $V_{\mathbb{Z}^d}(n) \approx n^d$.
- Recall that the Heisenberg group $\mathbb{H}(\mathbb{Z})$ is the 2-step torsion-free nilpotent group that can be defined as the group of triples $(x, y, z) \in \mathbb{Z}^3$ equipped with the group operation

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx').$$

Growth function

Let Λ be a group generated by a finite subset S. Define the growth function of Λ

$$V_\Lambda(n) = |S^n| = \left\{g \in \Lambda \mid g = s_1^{\pm 1} \dots s_n^{\pm 1},\ s_i \in S\right\}.$$

The asymptotic behavior of V_G does not depend on S.

Exemples

- $V_{\mathbb{Z}^d}(n) \approx n^d$.
- Recall that the Heisenberg group $\mathbb{H}(\mathbb{Z})$ is the 2-step torsion-free nilpotent group that can be defined as the group of triples $(x, y, z) \in \mathbb{Z}^3$ equipped with the group operation

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx').$$

$V_{\mathbb{H}(\mathbb{Z})}(n) \approx n^4$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$. More specifically:

Theorem (Bowen/Delabie-Koivisto-Le Maître-T 20)

If there exists a (φ, L^0)-OE from Λ to Γ, then $V_\Lambda \circ \varphi \preccurlyeq V_\Gamma$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$. More specifically:

Theorem (Bowen/Delabie-Koivisto-Le Maître-T 20)

If there exists a (φ, L^0)-OE from Λ to Γ, then $V_\Lambda \circ \varphi \preccurlyeq V_\Gamma$.

- Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$. More specifically:

Theorem (Bowen/Delabie-Koivisto-Le Maître-T 20)

If there exists a (φ, L^0)-OE from Λ to Γ, then $V_\Lambda \circ \varphi \preccurlyeq V_\Gamma$.

- Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.
- If Λ has exponential growth and if Λ and $\mathbb{Z}$ are are φ-OE, then $\varphi(n) \lesssim \log n$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$. More specifically:

Theorem (Bowen/Delabie-Koivisto-Le Maître-T 20)

If there exists a (φ, L^0)-OE from Λ to Γ, then $V_\Lambda \circ \varphi \preccurlyeq V_\Gamma$.

- Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.
- If Λ has exponential growth and if Λ and $\mathbb{Z}$ are are φ-OE, then $\varphi(n) \lesssim \log n$.

Theorem (Austin 16)

If $\Lambda = \mathbb{Z}^d$, and if Γ is L^1-OE to Λ, then Γ is virtually $\mathbb{Z}^d$.

Growth function

Theorem (Bowen 16)

If Λ and Γ are L^1-OE, then $V_\Lambda \approx V_\Gamma$.

Hence $\mathbb{Z}^d$ is L^1-0E to $\mathbb{Z}^{d'}$, then $d = d'$. More specifically:

Theorem (Bowen/Delabie-Koivisto-Le Maître-T 20)

If there exists a (φ, L^0)-OE from Λ to Γ, then $V_\Lambda \circ \varphi \preccurlyeq V_\Gamma$.

- Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $\mathbb{Z}^{d+k}$ to $\mathbb{Z}^d$, then $p \leq d/(d+k)$.
- If Λ has exponential growth and if Λ and $\mathbb{Z}$ are are φ-OE, then $\varphi(n) \lesssim \log n$.

Theorem (Austin 16)

If $\Lambda = \mathbb{Z}^d$, and if Γ is L^1-OE to Λ, then Γ is virtually $\mathbb{Z}^d$.

For instance $\mathbb{H}(\mathbb{Z})$ and $\mathbb{Z}^4$ are not L^1-OE, although they have same growth.

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Remark

- *Λ is amenable iff $F\o l < \infty$. The general philosophy is: the* **faster** *$F\o l_\Lambda$ the* **less amenable** *is Λ.*

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Remark

- *Λ is amenable iff $F\o l < \infty$. The general philosophy is: the* **faster** *$F\o l_\Lambda$ the* **less amenable** *is Λ.*
- *Coulhon and Saloff-Coste inequality: $V \lesssim F\o l$, and in general, $F\o l$ is much faster than V.*

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \triangle A|}{|A|} \leq 1/n,\ \forall s \in S\right\}$$

Remark

- *Λ is amenable iff $F\o l < \infty$. The general philosophy is: the* **faster** *$F\o l_\Lambda$ the* **less amenable** *is Λ.*
- *Coulhon and Saloff-Coste inequality: $V \lesssim F\o l$, and in general, $F\o l$ is much faster than V.*

Exemples

- For $\mathbb{Z}^d$, $F\o l(n) \approx V(n) \approx n^d$.

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{ |A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n, \ \forall s \in S \right\}$$

Remark

- *Λ is amenable iff $F\o l < \infty$. The general philosophy is: the* **faster** *$F\o l_\Lambda$ the* **less amenable** *is Λ.*
- *Coulhon and Saloff-Coste inequality: $V \lesssim F\o l$, and in general, $F\o l$ is much faster than V.*

Exemples

- For $\mathbb{Z}^d$, $F\o l(n) \approx V(n) \approx n^d$.
- For $\mathbb{H}(\mathbb{Z})$, $F\o l(n) \approx V(n) \approx n^4$.

Quantify amenability: Følner profile

Definition

Let Λ be a group generated by a finite subset S. Define its Følner function

$$F\o l(n) = \min\left\{|A| \mid \frac{|As \vartriangle A|}{|A|} \leq 1/n, \ \forall s \in S\right\}$$

Remark

- *Λ is amenable iff $F\o l < \infty$. The general philosophy is: the* **faster** *$F\o l_\Lambda$ the* **less amenable** *is Λ.*
- *Coulhon and Saloff-Coste inequality: $V \lesssim F\o l$, and in general, $F\o l$ is much faster than V.*

Exemples

- For $\mathbb{Z}^d$, $F\o l(n) \approx V(n) \approx n^d$.
- For $\mathbb{H}(\mathbb{Z})$, $F\o l(n) \approx V(n) \approx n^4$.
- For lamplighter groups (Erschler 06): $F \wr \mathbb{Z}^d = \bigoplus_{\mathbb{Z}^d} F \rtimes \mathbb{Z}^d$, $F\o l(n) \approx e^{n^d}$.

Invariance of the Følner function

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $Føl_\Lambda \approx Føl_\Gamma$.*

Invariance of the Følner function

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If* Λ *and* Γ *are* L^1*-OE, then* $Føl_\Lambda \approx Føl_\Gamma$.
- *More generally, if* Λ *and* Γ *are* (φ, L^0)*-OE for some concave increasing function* φ*, then*

Invariance of the Følner function

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If* Λ *and* Γ *are* L^1*-OE, then* $Føl_\Lambda \approx Føl_\Gamma$.
- *More generally, if* Λ *and* Γ *are* (φ, L^0)*-OE for some concave increasing function* φ*, then*

$$Føl_\Lambda \circ \varphi \lesssim Føl_\Gamma.$$

Invariance of the Følner function

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If* Λ *and* Γ *are* L^1*-OE, then* $Føl_\Lambda \approx Føl_\Gamma$.
- *More generally, if* Λ *and* Γ *are* (φ, L^0)*-OE for some concave increasing function* φ*, then*

$$Føl_\Lambda \circ \varphi \lesssim Føl_\Gamma.$$

Corollary

Let $d, k \in \mathbb{N}$. *If there exists an* (L^p, L^0)*-OE from* $F \wr \mathbb{Z}^{d+k}$ *to* $F \wr \mathbb{Z}^d$*, then* $p \leq d/(d+k)$.

Invariance of the Følner function

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *If Λ and Γ are L^1-OE, then $Føl_\Lambda \approx Føl_\Gamma$.*
- *More generally, if Λ and Γ are (φ, L^0)-OE for some concave increasing function φ, then*
$$Føl_\Lambda \circ \varphi \lesssim Føl_\Gamma.$$

Corollary

Let $d, k \in \mathbb{N}$. If there exists an (L^p, L^0)-OE from $F \wr \mathbb{Z}^{d+k}$ to $F \wr \mathbb{Z}^d$, then $p \leq d/(d+k)$.

Corollary (No quantitative version of OW's theorem)

For all Λ amenable, and all increasing unbounded φ, there exists another amenable group Γ that **is not** *φ-OE to Λ.*

Based on constructions of Brieussel-Zheng (2021).

What about a converse?

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.

What about a converse?

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.
- $\mathbb{Z}^4$ *and* $\mathbb{H}(\mathbb{Z})$ *are* L^p*-OE for all* $p < 1$.

What about a converse?

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.
- $\mathbb{Z}^4$ *and* $\mathbb{H}(\mathbb{Z})$ *are* L^p*-OE for all* $p < 1$.
- *The lamplighter group and* $\mathbb{Z}$ *are* $\log n^{1-\varepsilon}$*-OE for all* $\varepsilon > 0$.

New method of **Explicit construction of OE-couplings** for a given pair of **amenable** groups.

What about a converse?

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.
- $\mathbb{Z}^4$ *and* $\mathbb{H}(\mathbb{Z})$ *are* L^p*-OE for all* $p < 1$.
- *The lamplighter group and* $\mathbb{Z}$ *are* $\log n^{1-\varepsilon}$*-OE for all* $\varepsilon > 0$.

New method of **Explicit construction of OE-couplings** for a given pair of **amenable** groups.

Let us explain it for $\mathbb{Z}$ and $\mathbb{Z}^2$.

What about a converse?

The previous result are nearly optimal in a number of situation. For instance

Theorem (Delabie-Koivisto-Le Maître-Tessera 20)

- *Let* $d, k' \in \mathbb{N}$. *Then* $\mathbb{Z}^d$ *and* $\mathbb{Z}^{d+k}$ *are* L^p*-OE for all* $p < d/(d+k)$.
- $\mathbb{Z}^4$ *and* $\mathbb{H}(\mathbb{Z})$ *are* L^p*-OE for all* $p < 1$.
- *The lamplighter group and* $\mathbb{Z}$ *are* $\log n^{1-\varepsilon}$*-OE for all* $\varepsilon > 0$.

New method of **Explicit construction of OE-couplings** for a given pair of **amenable** groups.

Let us explain it for $\mathbb{Z}$ and $\mathbb{Z}^2$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as: $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\dots) = (1,0,0,1\dots)$
  $a \cdot (1,0,0,\dots) = (0,1,0,\dots)$
  $a \cdot (1,1,1,0,\dots) = (0,0,0,1,\dots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\dots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2**-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2**-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) =$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2**-odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4**-odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Two sequences belong to the **same orbit** if and only if they differ by at most finitely many coordinates.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

Preliminaries:

- **The** 2-**odometer:** consider the action of $\mathbb{Z}$ on the $\{0,1\}^{\mathbb{N}}$, defined as follows. The generator a of $\mathbb{Z}$ acts as:
  $a \cdot (0,0,0,1,\ldots) = (1,0,0,1\ldots)$
  $a \cdot (1,0,0,\ldots) = (0,1,0,\ldots)$
  $a \cdot (1,1,1,0,\ldots) = (0,0,0,1,\ldots)$
- **The** 4-**odometer:** : consider the action of $\mathbb{Z}$ on the $\{0,1,2,3\}^{\mathbb{N}}$, defined as follows. $a \cdot (1,2,0,3,\ldots) = (2,2,0,3,\ldots)$
  $a \cdot (3,1,2,0,\ldots) = (0,2,2,0,\ldots)$
  $a \cdot (3,3,3,3,1,0,\ldots) = (0,0,0,0,2,0,\ldots)$

These actions **preserve the product measure** on $\{0,1\}^{\mathbb{N}}$ and $\{0,1,2,3\}^{\mathbb{N}}$.

Two sequences belong to the **same orbit** if and only if they differ by at most finitely many coordinates.

The d-odometer is the action of $\mathbb{Z}$ by translation on $\mathbb{Z}_d$ (the ring of d-adic numbers).

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0, 1, 2, 3\}^{\mathbb{N}}$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

Constructing an OE between $\mathbb{Z}$ and $\mathbb{Z}^2$

The actions of $\mathbb{Z}$ and $\mathbb{Z}^2$:

- We let $\mathbb{Z}$ acts on the 4-odometer: $\{0,1,2,3\}^{\mathbb{N}}$
- We let $\mathbb{Z}^2$ acts on a product of 2-odometers: $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

The orbit equivalence: $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1,2,3\}^{\mathbb{N}}$ is defined

$$F(x,y) = x + 2y.$$

Example: if $x = (0,1,1,\ldots)$, $y = (1,0,1,\ldots)$, then

$$F(x,y) = (0+2, 1+0, 1+2, \ldots) = (2,1,3,\ldots).$$