Extended diffeomorphism groups for noncommutative manifolds[1]

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[1] This is a joint work with B. Ćaćić.

Outline

Introduction

Notations: Let B be a unital pre C^* algebra equipped with a $*$-exterior algebra (Ω_B, d_B) that is,

Definition

- A graded algebra $\Omega_B = \bigoplus_{n\geq 0} \Omega_B^n$, with $\Omega_B^0 = B$
- $d_B : \Omega_B^n \to \Omega_B^{n+1}$ s.t. $d_B^2 = 0$ and

$$d_B(\omega \wedge \rho) = (d_B\omega) \wedge \rho + (-1)^n \omega \wedge d_B\rho, \ \forall \omega, \rho \in \Omega_B \text{ and } \forall \omega \in \Omega_B^n$$

- $B, d_B B$ generate Ω_B
- There exists an antilinear involutive map $* : \Omega_B^n \to \Omega_B^n$ for all n such that

$$(d_B\xi)^* = d_B(\xi^*) \ \forall \xi \in \Omega_B$$
$$(\xi \wedge \eta)^* = (-1)^{nm} \eta^* \wedge \xi^* \ \forall \xi \in \Omega_B^n, \forall \eta \in \Omega_B^m$$

Extended diffeomorphism group

Further, let's denote,

1. $(\Omega^1_B)_{sa} = \{\omega \in \Omega_B \mid \omega^* = \omega\}$
2. $[\cdot,\cdot]$ to be the supercommutator in Ω_B with respect to pairity of degree.
3. $Aut(\Omega_B)$ denotes the automorphism group of the graded algebra Ω_B.
4. $Aut_0(\Omega_B) := \left\{\varphi \in Aut(\Omega_B) \mid \varphi_{\restriction_B} = id_B\right\}$

Since, inner automorphisms of B aren't *naive* i.e. φ doesn't commute with d_B we seek the following generalization.

Definition

The *extended diffeomorphism group* of B with respect to (Ω_B, d_B), denoted by $\widetilde{Diff}(B)$, is defined to be the subgroup

$$\left\{(\omega,\varphi) \in (\Omega^1_B)_{sa} \rtimes Aut(\Omega_B) \mid \forall \beta \in \Omega_B, d(\beta) - \varphi \circ d \circ \varphi^{-1}(\beta) = \mathbf{i}[\omega,\beta]\right\}$$

Furthermore, we define, $\widetilde{Diff}_0(B) := \{(\omega,\varphi) \in \widetilde{Diff}(B) \mid \varphi \in Aut_0(\Omega_B)\}$ whose elements are said to be *topologically trivial*.

Why?

1. The most conservative notion of diffeomorphism of a noncommutative manifold that includes all inner automorphisms.
2. Computation of $\mathrm{DPic}(B)$.
3. Computation of moduli spaces of solutions to Euclidean Maxwell's equations.

Examples

1. Let X be a closed manifold. $B = C^\infty(X)$ with sup norm and (Ω_B, d_B) to be the de Rham calculus.
2. Irrational noncommutative 2-Torus, $\mathcal{A}_\theta^\infty$.
3. Algebraic standard Podleś sphere, $\mathcal{O}_q(\mathbb{C}P^1)$.

Statement of the problem

Let $\pi : \widetilde{Diff}(B) \to Aut(B)$ denote the projection map,

$$\pi(\omega, \varphi) := \varphi_{\restriction B}$$

We have the short exact sequence,

$$1 \longrightarrow \widetilde{Diff}_0(B) \longrightarrow \widetilde{Diff}(B) \xrightarrow{\pi} Aut(B) \longrightarrow 1$$

Q1. Does this short exact sequence split?

Q2. Can we explicitly compute the groups $\widetilde{Diff}_0(B)$ and $\widetilde{Diff}(B)$?

Some useful results

Theorem (Elliott)

$Aut(\mathcal{A}_\theta^\infty) \cong \left(\mathcal{U}(A_\theta^\infty)^0/\mathcal{U}(1)\right) \rtimes \left(\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})\right)$

Theorem (Krähmer)

We have a group isomorphism, $\alpha : U(1) \to Aut(\mathcal{O}_q(\mathbb{C}P^1))$ *given by,*

$$\alpha(z) := \lambda_z$$

where for each $z \in U(1)$*,*

$$\lambda_z(x_0) := x_0, \text{ and } \lambda_z(x_\pm) := z^\pm x_\pm$$

1. G.A. Elliott. "The diffeomorphism group of the irrational rotation C*-algebra". In: *C. R. Math. Rep. Acad. Sci. Canada* Vol. 8(5) (1986), pp. 329–334
2. Ulrich Krähmer. "On the Non-standard Podleś Spheres". In: *C*-algebras and Elliptic Theory II*. ed. by Dan Burghelea et al. Basel: Birkhäuser Basel, 2008, pp. 145–147

$\mathcal{O}_q(\mathbb{C}P^1)$

Definition

Let $q \in (0,1)$. The algebraic standard Podleś sphere is the $*$-algebra generated by elements x_0, x_+ and x_- subject to the relations,

$$x_0 x_\pm = q^{\pm 2} x_\pm x_0, \;\; x_\mp x_\pm = q^{\pm 2} x_0^2 + (1 + q^{\pm 2}) x_0, \;\; x_\pm^* = -q^{\pm 1} x_\mp$$

Due to a theorem by Majid, we have that, $(\Omega_q(\mathbb{C}P^1, d_q))$ is a $*$-FODC on $\mathcal{O}_q(\mathbb{C}P^1)$ where,

$$\Omega_q(\mathbb{C}P^1) = \mathcal{L}_{-2} e^+ \oplus \mathcal{L}_2 e^-$$

and $d_q : \mathcal{O}_q(\mathbb{C}P^1) \to \Omega_q(\mathbb{C}P^1)$ defined by, $d_q := d_{q,hor}\restriction_{\mathcal{O}_q(\mathbb{C}P^1)}$; arising from the $*$-FODC on $\mathcal{O}_q(SU(2))$ with $U(1)$ action.

Computing $\widetilde{Diff}_0(\mathcal{O}_q(\mathbb{C}P^1))$ and $\widetilde{Diff}(\mathcal{O}_q(\mathbb{C}P^1))$

To answer Q1, define, $\rho : U(1) \to \widetilde{Diff}(\mathcal{O}_q(\mathbb{C}P^1))$ by,

$$\rho(z) := (0, \lambda_z)$$

This gives us a split extension

$$\widetilde{Diff}(\mathcal{O}_q(\mathbb{C}P^1)) \cong \widetilde{Diff}_0(\mathcal{O}_q(\mathbb{C}P^1)) \rtimes U(1)$$

However, we have that $\widetilde{Diff}_0(\mathcal{O}_q(\mathbb{C}P^1))$, given by

$$\left\{ (p,s) \in \mathcal{O}_q(SU(2))_{\pm 2} \rtimes \mathbb{C}^{x} \;\middle|\; \forall b \in \mathcal{O}_q(\mathbb{C}P^1), \begin{array}{c} (1-s)\partial_+(b) = i[p,b] \\ \text{and} \\ (1-\bar{s})\partial_-(b) = i[-q^{-1}p^*, b] \end{array} \right\}$$

is trivial and hence $\widetilde{Diff}(\mathcal{O}_q(\mathbb{C}P^1)) \cong U(1)$

$\mathcal{A}_\theta^\infty$

Definition

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The irrational NC 2-torus is the $*$-algebra of rapidly decaying Laurent series in the generators u and v satisfying

$$vu = e^{2\pi \mathbf{i}\theta} uv$$

This comes equipped with the graded $*$-algebra $\Omega_\theta(\mathbb{T}^2)$ over $\mathcal{A}_\theta^\infty$, generated by central self adjoint elements $e_1, e_2 \in \Omega_\theta^1(\mathbb{T}^2)$ satisfying,

$$(e^1)^2 = (e^2)^2 = e^1e^2 + e^2e^1 = 0$$

and $d : \mathcal{A}_\theta^\infty \to \Omega_\theta(\mathbb{T}^2)$ defined by,

$$d(a) = \delta_1(a)e^1 + \delta_2(a)e^2$$

where, for all $m, n \in \mathbb{Z}$,

$$\delta_1(u^m v^n) = 2\pi i m u^m v^n, \delta_2(u^m v^n) = 2\pi i n u^m v^n, de^1 = de^2 = 0$$

Computing $\widetilde{Diff}_0(\mathcal{A}_\theta^\infty)$ and $\widetilde{Diff}(\mathcal{A}_\theta^\infty)$

To answer Q1, define, $\rho : \left(\mathcal{U}(A_\theta^\infty)^0/\mathcal{U}(1)\right) \rtimes \left(\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})\right) \to \widetilde{Diff}(A_\theta^\infty)$ by,

$$\rho([w],(z_1,z_2),g)) := \rho_1([w]) \cdot \rho_2((z_1,z_2)) \cdot \rho_3(g)$$

where,

$$[w] \mapsto (-id(w)w^*, Ad_{[w]}) : \left(\mathcal{U}(A_\theta^\infty)^0/\mathcal{U}(1)\right) \xrightarrow{\rho_1} \widetilde{Diff}(A_\theta^\infty)$$
$$(z_1,z_2) \mapsto (0,\alpha_{(z_1,z_2)}) : \mathbb{T}^2 \xrightarrow{\rho_2} \widetilde{Diff}(A_\theta^\infty)$$
$$g \mapsto (0,\sigma_g) : SL(2,\mathbb{Z}) \xrightarrow{\rho_3} \widetilde{Diff}(A_\theta^\infty)$$

then ρ is a group homomorphism that splits the short exact sequence.

Theorem

$$\widetilde{Diff}_0(A_\theta^\infty) \cong \mathbb{R}^2$$
$$\widetilde{Diff}(A_\theta^\infty) \cong \mathbb{R}^2 \rtimes \left[\left(\mathcal{U}(A_\theta^\infty)^0/\mathcal{U}(1)\right) \rtimes \left(\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})\right)\right]$$

Computing $\widetilde{Diff}_0(\mathcal{A}_\theta^\infty)$and $\widetilde{Diff}(\mathcal{A}_\theta^\infty)$

Contd.

Proof.

We have that,

$$\widetilde{Diff}(A_\theta^\infty) \cong \widetilde{Diff}_0(A_\theta^\infty) \rtimes \left[\left(\mathcal{U}(A_\theta^\infty)^0/\mathcal{U}(1)\right) \rtimes \left(\mathbb{T}^2 \rtimes SL(2,\mathbb{Z})\right)\right]$$

However, by a result of Bratelli-Elliott-Jorgenson[3], we see that the topologically trivial elements of $\widetilde{Diff}(A_\theta^\infty)$ are of the form

$$\{(\omega, id) \mid \omega \in Z(\Omega_\theta^1(\mathbb{T}^2)_{sa})\} = Z(\Omega_\theta^1(\mathbb{T}^2)_{sa}) \times \{id\} \cong \mathbb{R}^2$$

and we have the theorem. □

3. P. E. T. Jorgensen, G. A. Elliott, and O. Bratteli. "Decomposition of unbounded derivations into invariant and approximately inner parts.". In: *Journal für die reine und angewandte Mathematik* 346 (1984), pp. 166–193

Thank you!