Quantum superchannels on the space of quantum channels

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  for all matrices ρ

- The requirements that quantum systems combine using tensor products, and that the identity map is a valid channel is what implies the *completely positive* (CP) condition.

Definition 1.1

A *quantum channel* is a linear CPTP map $\phi : M_d \to M_r$.

Choi matrix

- Let $E_{i,j}$, $1 \leq i, j \leq d$ denote the matrix units in M_d. We have an isomorphism between linear maps and block matrices via $\phi \mapsto (\phi(E_{i,j})) = \sum_{i,j} E_{i,j} \otimes \phi(E_{i,j})$..

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- The matrix $C_\phi := (\phi(E_{i,j}))$ is called the *Choi matrix* or Choi-Jamiołkowski matrix of the map.

$$C_\phi = \begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) & \cdots & \phi(E_{1d}) \\ \phi(E_{21}) & \phi(E_{22}) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \phi(E_{d1}) & \cdots & \cdots & \phi(E_{dd}) \end{pmatrix}$$

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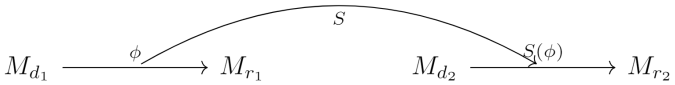
- Choi's theorem says that a linear map $\phi : M_d \to M_r$ is completely positive if and only if $C_\phi \geq 0$ in $M_d(M_r)$.

Superchannel

Definition 1.2

A *quantum superchannel* is a linear map
$S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ which satisfies

- CP preserving: S sends CP maps to CP maps
- Completely CP preserving: For any d, r if $\text{id}_{d,r}$ is the identity map acting on $\mathcal{L}(M_d, M_r)$ then $S \otimes \text{id}_{d,r}$ is CP preserving
- TP preserving: S sends TP maps to TP maps



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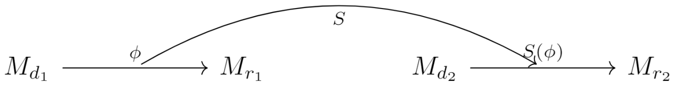

- Superchannels give an induced map on Choi matrices

$$\widetilde{S}(C_\phi) = C_{S(\phi)}$$

.

Characterisation theorem

Theorem 1.1 (Chiribella, D'Ariano, Perinotti; 2008)

If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a quantum superchannel then there exists two quantum channels ψ_{pre}, ψ_{post} such that

$$S(\phi) = \psi_{post} \circ (id_e \otimes \phi) \circ \psi_{pre}$$

where e is the dimension of an auxilliary space.

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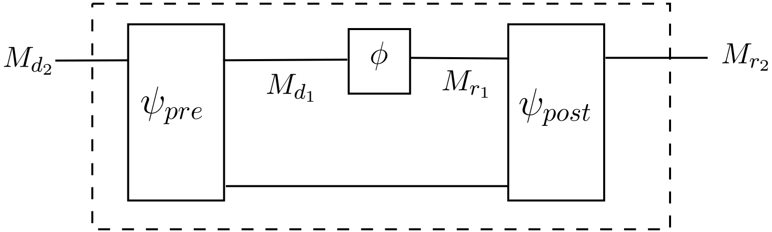

Minimal dilation dimension

- Partial Trace: $\mathrm{Tr}_2 = \mathsf{id} \otimes \mathrm{Tr}$ for example,

$$\mathrm{Tr}_2(A \otimes B) = A \cdot \mathrm{Tr}(B)$$

- In the characterisation of superchannels

$$S(\phi) = \psi_{\mathsf{post}} \circ (\phi \otimes \mathsf{id}_e) \circ \psi_{\mathsf{pre}}$$

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- In a 2020 paper Gour and Scandolo show that this is unique in the sense that any other characterisation with equal or smaller dimension is equivalent up to action by a unitary channel.

Space of Quantum Channels

Definition 2.1

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Let $S(d,r) \subset M_d(M_r)$ be the set of block matrices $(P_{i,j})$ such that for all $1 \leq i,j \leq d$, $\mathrm{Tr}(P_{i,i}) = \mathrm{Tr}(P_{j,j})$ and for $i \neq j$ $\mathrm{Tr}(P_{i,j}) = 0$.

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Theorem 2.1

$S(d,r)$ is an operator system and is completely order isomorphic to $SCPTP(d,r)$ via the Choi map $\phi \mapsto C_\phi$.

QSC

Definition 2.3

Given two spaces of quantum channels $SCPTP(d_i, r_i)$, $i = 1, 2$, a *QSC* is a linear map $\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ which satisfies

1. if ϕ is CPTP then $\Gamma(\phi)$ is CPTP
2. given any other $d_3, r_3 \in \mathbb{N}$ and the identity map $\mathrm{id}_{d_3,r_3} : SCPTP(d_3, r_3) \to SCPTP(d_3, r_3)$ then $\Gamma \otimes \mathrm{id}_{d_3,r_3} : SCPTP(d_1, r_1) \otimes SCPTP(d_3, r_3) \to SCPTP(d_2, r_2) \otimes SCPTP(d_3, r_3)$ sends CPTP maps to CPTP maps.

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A QSC induces a map $\widetilde{\Gamma} : S(d_1, r_1) \to S(d_2, r_2)$ via

$$C_\phi \mapsto C_{\Gamma(\phi)}$$

Problem with QSC's

- We have an inclusion

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  - Can still recover by extending

Extending QSC's

Theorem 2.2

If $\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ *preserves CPTP maps then it is a QSC if and only if* $\widetilde{\Gamma}$ *is completely positive.*

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Theorem 2.3

Every QSC extends to a quantum superchannel.

Use Arvesons extension theorem on the (completely positive) induced map. Then Choi's theorem will guarantee its associated map is "completely CP preserving".

- Is the extension unique?

QSC vs superchannel

- For some maps the extension from a QSC to a superchannel is unique e.g. the Choi matrix of a Schur product superchannel is fixed by its action on quantum channels.
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- For some maps the extension from a QSC to a superchannel is unique e.g. the Choi matrix of a Schur product superchannel is fixed by its action on quantum channels.
- However in general the extension of a QSC is not unique
- Therefore many different quantum super channels can restrict to the same QSC. Thus, if we are really only concerned with the action of a superchannel on quantum channels, then we are really only concerned with the corresponding QSC.
- This also has implications for the characterisation theorem

Choi matrix of equivalent extensions

QSC's aren't defined on all the basis matrix elements, e.g. diagonals $E_{i,i}$ but they are defined on certain sums of them

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Theorem 2.4

If S_1 and S_2 are superchannels extending the same QSC then

$$\mathrm{Tr}_{d_1} C_{\widetilde{S}_1 - \widetilde{S}_2} = 0$$

- Essentially, there isn't enough quantum channels to determine what the entries of $C_{\widetilde{S}}$ should be, but there are enough to determine how the blocks sum.

Auxilliary dimension of different extensions

- If superchannels S_1, S_2 extend the same QSC, another equivalent extension is given by any convex combination $S = p_1 S_1 + p_2 S_2$ for $p_1, p_2 > 0$, $p_1 + p_2 = 1$.

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- In many examples it seems that extreme points of the set of extensions have characterisation with minimal auxilliary dimension e.
- If $U = U_1 \otimes U_2$ where $U_1 \in \mathcal{U}(d)$ and $U_2 \in \mathcal{U}(r)$ then $\widetilde{S}(C_\phi) = U C_\phi U^*$ is a superchannel

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  - In fact every superchannel given by unitary conjugation must be of this form. and they are extreme points of the set of extensions of their QSC and always have auxilliary dimension $e = 1$.

Extreme points of sets of CP maps

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- Let $\mathcal{S} \subset M_n$ and let $\mathcal{T} \subset M_m$. For a CP $\phi : M_n \to M_m$ define the convex set $CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \phi]$ to be CP maps from M_n to M_m equal ϕ on $\mathcal{S}$ and duals equal ϕ^* on $\mathcal{T}$

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Theorem 2.5

$\phi \in CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \Phi]$ is extreme iff it has an expression $\phi(A) = \sum_i V_i^* A V_i$ such that for any self-adjoint spanning sets $\{A_k\}_k$ for $\mathcal{S}$ and $\{B_l\}_l$ for $\mathcal{T}$ the following set is linearly independent:

$$\{\bigoplus_k V_i^* A_k V_j \bigoplus_l V_j B_l V_i^*\}_{ij}$$

.

UCP maps

- Let $r_1 = r_2 = 1$. Since $S(d,1) = \text{span}\{I_d\}$, any unital completely-positive (UCP) map $\widetilde{S} : M_{d_1} \to M_{d_2}$ is a superchannel and all these maps define the same QSC.

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- Fix $d_1 = d_2 = 3$. The anti-symmetric *Werner-Holevo* channel is given by the map $\phi : M_3 \to M_3$

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$$\phi(\rho) = \frac{\mathrm{Tr}[\rho] I_3 - \rho^T}{2}.$$

- Can show it is an extreme point of the set of extensions despite having a Choi-Kraus rank of 3.

TP extension not always possible

- A QSC preserves the "trace scaling factor" of its input map. On Choi matrices it sends block matrices of trace λd_1 to block matrices of trace λd_2. Thus the map

$$\frac{d_1}{d_2}\widetilde{\Gamma}$$

is a CPTP map.

- If we extend $\widetilde{\Gamma}$ to a superchannel $\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ can we do so in a way such that $\widetilde{S}$ is also TP?
- No, found an example of a QSC which cannot be extended to give a TP map.

Tensor of QSCs depends on extension

- Take QSC's $\Gamma_1 : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ and $\Gamma_2 : SCPTP(d_3, r_3) \to SCPTP(d_4, r_4)$. Extend each to a superchannel S_1, S_2 respectively. Then $S_1 \otimes S_2$ is a superchannel on the combined spaces and it restricts to give a QSC on $SCPTP(d_1 d_2, r_1 r_2)$. This is not necessarily unique.
- For example, if $S_a\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = a$ and $S_b\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = b$ then S_a and S_b extend the same QSC. Take input Choi matrix to be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in S(4, 2)$$

  Then for any other superchannel S, $S \otimes S_a$ and $S \otimes S_b$ give different QSC's.

Wigner's theorem

- Wigner's theorem says that any bijective map, $\mathcal{H} \to \mathcal{H}$, on a Hilbert space that preserves the *transition probability* $|\langle\phi|\psi\rangle|$ between any two vectors is given by a unitary or anti-unitary map.
- An equivalent version of Wigner's theorem can be given in terms of density matrices. Let $\mathcal{D}(\mathcal{H})$ be the set of density matrices acting on $\mathcal{H}$. A *state space symmetry* is a bijective map $S : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ which satisfies

$$S(p\rho+(1-p)\sigma) = pS(\rho)+(1-p)S(\sigma), \quad \forall\rho,\sigma \in \mathcal{D}(\mathcal{H}), p \in [0,1]$$

- Wigner's theorem then says that every state space symmetry is given by either a unitary map $\rho \mapsto U\rho U^*$ or anti-unitary map $\rho \mapsto U\rho^T U^*$, where $U \in \mathcal{B}(\mathcal{H})$ is a unitary.

Quantum operation symmetries

Define a quantum *operation symmetry* to be an invertible linear map $S : \mathcal{L}(M_d, M_r) \to \mathcal{L}(M_d, M_r)$ which preserves the set of completely-positive trace non-increasing maps.

Theorem 3.1 (G Chiribella, E Aurell, K Życzkowski, 2021)

If S is an operation symmetry then

$$S(\phi) = S_{post} \circ \phi \circ S_{pre}.$$

where S_{post} and S_{pre} are state space symmetries (both either unitary or anti-unitary).

Quantum channel symmetries

- Similarly we can define a quantum *channel symmetry* to be a bijective linear map

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which preserves the set of quantum channels.

- However, even if S is an invertible QSC it's extension might not be an operation symmetry.
- Positive extensions of channel symmetries will preserve CP trace non-increasing maps.
- Not all positive maps on operator systems have positive extensions

Thanks!

Thanks for listening!