Higher order moments and free cumulants of complex Wigner matrices

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Joint work with James A. Mingo.
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Outline

1. The Wigner ensemble
2. The moments and free cumulants of the Wigner ensemble
3. Partitioned permutations
4. Second order free cumulants
5. Third order free cumulants
6. Higher order case

FIELDS

Wigner ensemble

Definition

By a complex Wigner matrix X_N we mean a $N \times N$ random matrix of the form $X_N = \frac{1}{\sqrt{N}}(x_{i,j})$ such that,

- the entries are complex random variables;
- the matrix is self-adjoint: $x_{i,j} = \overline{x_{j,i}}$;
- all entries on and above the diagonal are independent: $\{x_{i,j}\}_{i<j} \cup \{x_{i,i}\}_i$ are independent;
- the entries above the diagonal, $\{x_{i,j}\}_{i<j}$, are identically distributed;
- the diagonal entries, $\{x_{i,i}\}_i$, are identically distributed;
- $\mathrm{E}(x_{i,j}) = 0$ for all i, j,
- $\mathrm{E}(x_{i,j}^2) = 0$ for all $i \neq j$,
- $\mathrm{E}(|x_{i,j}|^2) = 1$ for all i, j,
- $\mathrm{E}(|x_{i,j}|^k) < \infty$ for all i, j, k.

A collection $X = (X_N)_N$ of Wigner matrices satisfying these conditions will be called Wigner ensemble.

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Wigner's semi-circle law

Theorem (Wigner 1955)

$$\frac{1}{N}\lim_{N\to\infty} E(\mathrm{Tr}(X_N^m)) = \int_{\mathbb{R}} t^m d\mu(t),$$

where μ is the semicircle distribution on $[-2, 2]$.

Equivalently, if we let $\alpha_m := \frac{1}{N}\lim_{N\to\infty} E(\mathrm{Tr}(X_N^m))$. Then,

$$\alpha_m = \begin{cases} \frac{1}{m/2+1}\binom{m}{m/2} & \text{for } m \text{ even} \\ 0 & \textit{otherwise} \end{cases}$$

$$\sim$$

$$\alpha_m = |NC_2(m)|.$$

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Moment-cumulant relation

$$\alpha_m = \sum_{\pi \in NC(m)} \kappa_\pi ,$$

where,

$$\kappa_\pi = \prod_{B \subset \pi} \kappa_{|B|} .$$

Two different approaches

$$\alpha_m = |NC_2(m)|.$$

$$\kappa_m = \begin{cases} 1 & \text{if } m = 2 \\ 0 & \textit{otherwise} \end{cases}$$

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Fluctuation moments

What about the fluctuation moments?

$$\alpha_{m,n} := \lim_{N\to\infty} K_2(\mathrm{Tr}(X_N^m), \mathrm{Tr}(X_N^n)).$$

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Theorem (Male, Mingo, Péché, Speicher, '20)

For $m_1, m_2 \in \mathbb{N}$

$$\alpha_{m_1,m_2} = |NC_2(m_1, m_2)| + K_4|NC_2^{(2)}(m_1, m_2)|, \quad (1)$$

where $K_4 = K_4(x_{1,2}, x_{1,2}, x_{2,1}, x_{2,1})$ is the fourth classical cumulant of an off-diagonal element.

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What is the analogous statement in terms of cumulants?
What are the cumulants of order two equal to?

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Partitioned permutations

A partitioned permutation is a pair $(\mathcal{V}, \pi)$ consisting of $\pi \in S_n$ and $\mathcal{V} \in \mathcal{P}(n)$ with $\pi \leq \mathcal{V}$. The set of partitioned permutations is denoted by $\mathcal{PS}_n$. We let,

$$|(\mathcal{V}, \pi)| = 2|\mathcal{V}| - |\pi|,$$

with $|\mathcal{V}| = n - \#(\mathcal{V})$ and $|\pi| = n - \#(\pi)$. It is satisfied,

$$|(\mathcal{V} \vee \mathcal{U}, \pi\sigma)| \leq |(\mathcal{V}, \pi)| + |(\mathcal{U}, \sigma)|.$$

For $(\mathcal{V}, \pi), (\mathcal{W}, \sigma) \in \mathcal{PS}_n$ we define their product as,

$$(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = \begin{cases} (\mathcal{V} \vee \mathcal{W}, \pi\sigma) & \text{if } |(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| = |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)|, \\ 0 & \text{otherwise} \end{cases}$$

Definition

For $(\mathcal{U},\gamma) \in \mathcal{PS}_n$ fixed we say that $(\mathcal{V},\pi) \in \mathcal{PS}_n$ is $(\mathcal{U},\gamma)$-non crossing if,

$$(\mathcal{V},\pi)\cdot(0_{\pi^{-1}\gamma},\pi^{-1}\gamma) = (\mathcal{U},\gamma).$$

The set of $(\mathcal{U},\gamma)$-non crossing partitioned permutations will be denote by $\mathcal{PS}_{NC}(\mathcal{U},\gamma)$.

Let $m_1,\dots,m_r \in \mathbb{N}$ and

$$\gamma_{m_1,\dots,m_r} := (1,\dots,m_1)\cdots(m_1+\cdots+m_{r-1}+1,\dots,m),$$

with $m = \sum_{i=1}^r m_i$. We use the notation,

$$\mathcal{PS}_{NC}(m_1,\dots,m_r) := \mathcal{PS}_{NC}(1_m,\gamma_{m_1,\dots,m_r}).$$

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$$\mathcal{PS}_{NC}(m) = \{(0_\pi, \pi) : \pi \in NC(m)\} \cong NC(m).$$

$$\mathcal{PS}_{NC}(1_{m_1+m_2}, \gamma_{m_1,m_2}) = \{(0_\pi, \pi) \mid \pi \in S_{NC}(m_1, m_2)\}$$
$$\cup \{(\mathcal{V}, \pi) \mid \pi \in NC(m_1) \times NC(m_2), \mathcal{V} \vee \gamma = 1_n \text{ and } |\mathcal{V}| = |\pi| + 1\}$$

In the first part we have $S_{NC}(m_1, m_2)$ and in the second part $\mathcal{PS}_{NC}(m_1, m_2)'$. We shall write

$$\mathcal{PS}_{NC}(m_1, m_2) = S_{NC}(m_1, m_2) \cup \mathcal{PS}_{NC}(m_1, m_2)'.$$

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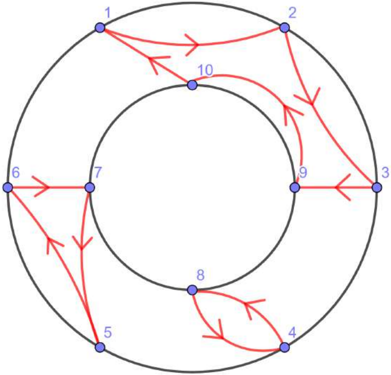
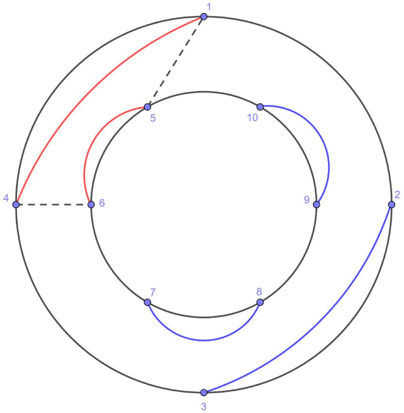
Partitioned permutations on two circles



$S_{NC}(6,4)$



$\mathcal{PS}_{NC}(4,6)'$

Second order cumulants

Definition

Given

$$\{\alpha_m\}_{m=1}^{\infty}, \text{ and } \{\alpha_{m_1,m_2}\}_{m_1,m_2=1}^{\infty}$$

a sequence of first and second order moments, we define the first $\{\kappa_m\}_m$, and second $\{\kappa_{m_1,m_2}\}_{m_1,m_2}$ order cumulants as the sequences given by the recursive formulas,

$$\alpha_m = \sum_{\pi \in NC(m)} \kappa_\pi \tag{2}$$

$$\alpha_{m_1,m_2} = \sum_{(\mathcal{U},\pi) \in \mathcal{PS}_{NC}(m_1,m_2)} \kappa_{(\mathcal{U},\pi)} \tag{3}$$

with $\kappa_{(\mathcal{U},\pi)}$ defined as follows,

$$\kappa_{(\mathcal{U},\pi)} = \prod_{\substack{D \text{ blocks of } \mathcal{U} \\ B_1,\dots,B_l \text{ cycles of } \pi \\ B_i \subset D}} \kappa_{|B_1|,\dots,|B_l|}$$

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Two different approaches

$$\alpha_{m_1,m_2} = |NC_2(m_1, m_2)| + K_4|NC_2^{(2)}(m_1, m_2)|.$$

$$\kappa_{p,q} = \begin{cases} 2K_4 & \textit{if} \quad p = q = 2 \\ 0 & \textit{otherwise} \end{cases}$$

Definition

For each $r \in \mathbb{N}$ let $(\alpha_{m_1,\ldots,m_r})_{m_1,\ldots,m_r=1}^{\infty}$ be a sequence indexed by r subscripts. We call this a moment sequence of order r. Given the moment sequences of orders at most r:

$$(\alpha_m)_{m=1}^{\infty}, (\alpha_{m_1,m_2})_{m_1,m_2=1}^{\infty}, \ldots, (\alpha_{m_1,\ldots,m_r})_{m_1,\ldots,m_r=1}^{\infty}$$

we define the free cumulants sequences

$$(\kappa_m)_{m=1}^{\infty}, (\kappa_{m_1,m_2})_{m_1,m_2=1}^{\infty}, \ldots, (\kappa_{m_1,\ldots,m_r})_{m_1,\ldots,m_r=1}^{\infty}$$

associated to these moment sequences with the recursive equations:

$$\alpha_{m_1,\ldots,m_t} = \sum_{(\mathcal{U},\pi)\in\mathcal{PS}_{NC}(m_1,\ldots,m_t)} \kappa_{(\mathcal{U},\pi)} \tag{4}$$

for $t = 1, \ldots, r$. With $\kappa_{(\mathcal{U},\pi)}$ is defined as follows:

$$\kappa_{(\mathcal{U},\pi)} = \prod_{\substack{B \text{ block of } \mathcal{U} \\ V_1,\ldots,V_i \text{ cycles of } \pi \text{ with } V_i \subset B}} \kappa_{|V_1|,\ldots,|V_i|}$$

The numbers $\kappa_{m_1,\ldots,m_r}$ are called the *free cumulants of order r* and the sequence $(\kappa_{m_1,\ldots,m_r})_{m_1,\ldots,m_r=1}^{\infty}$ is called the *free cumulant sequence of order r*.

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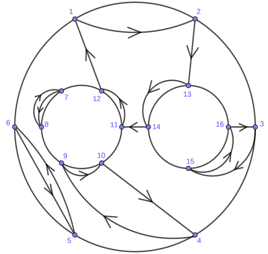
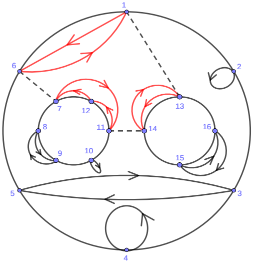
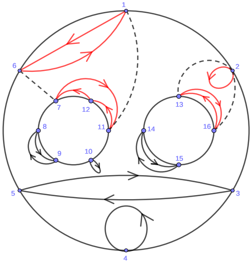


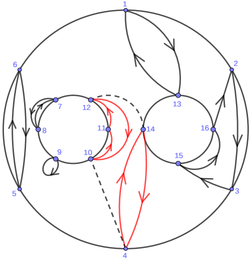
$S_{NC}(m_1, m_2)$



$\mathcal{PS}_{NC}^{(1,1,1)}(m_1, m_2, m_3)$



$\mathcal{PS}_{NC}^{(2,1,1)}(m_1, m_2, m_3)$



$\mathcal{PS}_{NC}^{(1,1)}(m_1, m_2, m_3)$

Third order case

Let X_N be a $N \times N$ Wigner matrix which satisfies the extra condition:

$$K_3(x_{1,1}, x_{1,1}, x_{1,1}) = 0.$$

Let,

$$\alpha_{m_1,m_2,m_3} := NK_3(\mathrm{Tr}(X_N^{m_1}), \mathrm{Tr}(X_N^{m_2}), \mathrm{Tr}(X_N^{m_3})).$$

Theorem (M, Mingo, '22)

For $m_1, m_2, m_3 \in \mathbb{N}$

$$\begin{aligned}\alpha_{m_1,m_2,m_3} = |NC_2(m_1, m_2, m_3)| + 4K_6|\mathcal{PS}_{NC_2}^{(1,1,1)}(m_1, m_2, m_3)| \\ + 4K_4^2|\mathcal{PS}_{NC_2}^{(2,1,1)}(m_1, m_2, m_3)| + 2K_4|\mathcal{PS}_{NC_2}^{(1,1)}(m_1, m_2, m_3)| \\ + (\mathring{K}_4 - 2K_4)|\mathcal{PS}_{NC_{2,1,1}}^{(1,1,1)}(m_1, m_2, m_3)|\end{aligned}$$

equivalently,

$$\kappa_{m,n,p} = \begin{cases} 4K_6 & \text{if } m = n = p = 2 \\ K_4^0 - 2K_4 & \text{if } \{m, n, p\} = \{2, 1, 1\} \\ 0 & \text{otherwise} \end{cases}$$

whit $K_6 = K_6(x_{1,2}, x_{1,2}, x_{2,1}, x_{2,1})$, $K_4^0 = K_4^0(x_{1,1}, x_{1,1}, x_{1,1}, x_{1,1})$ *and* $K_4 = K_4(x_{1,2}, x_{1,2}, x_{2,1}, x_{2,1})$.

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- $x_{i,i} \sim N(0,1)$ for all i, this is; $x_{i,i}$ has standard normal distribution.
- For every $n \in \mathbb{N}$, $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ and $i \neq j$ we have that,

$$K_n(x_{i,j}^{(\epsilon_1)}, \ldots, x_{i,j}^{(\epsilon_{2n+1})}) = 0$$

  whenever either n is odd, or n is even but the number of ϵ_i which are 1 is different to the number of ϵ_i which are -1. With $x_{i,j}^{(1)} = x_{i,j}$ and $x_{i,j}^{(-1)} = x_{j,i}$.

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Let,

$$\alpha^{(N)}_{m_1,\ldots,m_r} := N^{r-2} K_r(\mathrm{Tr}(X_N^{m_1}), \ldots, \mathrm{Tr}(X_N^{m_r})),$$

and

$$\alpha_{m_1,\ldots,m_r} := \lim_{N\to\infty} \alpha^{(N)}_{m_1,\ldots,m_r}.$$

Lemma

The limit;

$$\lim_{N\to\infty} \alpha^{(N)}_{m_1,\ldots,m_r}$$

exist for any $r \in \mathbb{N}$ and $m_1, \ldots, m_r \in \mathbb{N}$. Moreover,

$$\alpha_{m_1,\ldots,m_r} = \sum_{\substack{\pi\in\mathcal{P}(m)\\ \#(\pi)-m/2+r-2=0}} \sum_{\substack{\tau\in\mathcal{P}(m)\\ \tau\vee\gamma=1_m\\ \tau\in\mathcal{P}_{bal}(\pi)}} K_\tau(\pi). \tag{5}$$

THANK YOU

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