Simplicity of crossed products by FC-hypercentral groups

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Intuition: contains $\{\sum_{\text{finite}} a_t \lambda_t \mid t \in G, a_t \in A\}$ as a dense subset, and

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The reduced crossed product $A \rtimes_r G$ is the unique norm completion such that $E(\sum a_t \lambda_t) = a_e$ is a faithful conditional expectation.

A brief history, part 1

Classical results on C*-simplicity can be summarized as follows:

Simplicity of $C_r^*(\mathbb{F}_2)$	Powers, 1975
... many like $C_r^*(\mathbb{F}_2)$ *...*	*... many ...*
$A \rtimes_r G$ is simple if $G \curvearrowright A$ properly outer, **but not if and only if**	Elliott, Kishimoto, Olesen-Pedersen (1978-1982)
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Simplicity of $C_r^*(G)$, arbitrary G	Breuillard, Kalantar, Kennedy, Ozawa (2017, 2017, 2020)
Simplicity of $C(X) \rtimes_r G$, arbitrary G	Kawabe, 2020
Simplicity of $A \rtimes_r G$ (A noncommutative), **partial results**	Kennedy-Schafhauser, 2020

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Theorem (Hamana)

Injective and G-injective envelopes always exist. Denoted $I(A)$ and $I_G(A)$.

The main idea in the modern proofs

In the **commutative** setting, the first main result is the following:

Theorem (Breuillard, Kalantar, Kawabe, Kennedy, Ozawa)

Assume $G \curvearrowright X$ is minimal. Let $I_G(C(X)) = C(Z)$. Then $C(X) \rtimes_r G$ is simple if and only if the action of G on Z is free.

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Is there a **noncommutative analogue** of freeness?

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Proposition (Kallman, Hamana)

Every $\alpha \in \mathrm{Aut}(A)$ *decomposes as* $\alpha = \alpha_1 \oplus \alpha_2$ *on* $Ap \oplus A(1-p)$, *where* p *is an* α*-invariant central projection, and* α_1 *is inner and* α_2 *is properly outer.*

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Example: if $C(Z)$ is monotone complete, then $\mathrm{Fix}(\alpha)$ is clopen. So

$$C(Z) \cong C(Z)p_{\mathrm{Fix}(\alpha)} \oplus C(Z)p_{\mathrm{Fix}(\alpha)^{\complement}} \quad \text{(inner and free parts)}$$

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Example (Finite-dimensional counterexample to converse)

Consider $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle u \rangle \times \langle v \rangle$ and $A = M_2$, with G acting by

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In fact, $A = I(A) = I_G(A)$.
The action of G on $I_G(A)$ is not properly outer, but $A \rtimes_r G \cong M_4$ is simple. Vanishing obstruction also cannot hold (easy to check manually).

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The mental health hazard:

- Characterizing everything in terms of A instead of $I(A)$.

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Without the last part, this would say "not simple/etc..." if and only if "not properly outer". But some invariance is necessary.

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- α-invariant nonzero ideal $J \triangleleft A$.
- Unitary $u \in U(M(J))$, where $M(J)$ is the multiplier algebra.

With $\|\alpha|_J - \mathrm{Ad}\, u\| < 2$.

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Assume A is separable, and we are in a situation from before. The following are equivalent:

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- $\alpha_t = \mathrm{Ad}\, u$ *on* $I(A)p$
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continued on next slide...

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  [I promise this makes sense even if *su* and *u* lie in different algebras]

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- FIN -