Interacting Particle Systems:

Zero-Range Interaction and Properties

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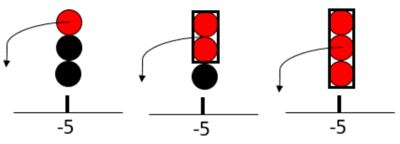

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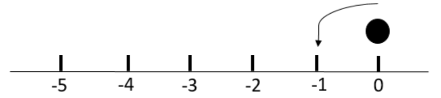
Section 1: Zero-range processes

Random Walk: single particle

Particle is positioned at 0 and after waiting an exponential amount of time it jumps one site to the left.

- Inter-jump time $\sim \mathrm{Exp}(\lambda)$
- Particle position at time $t \sim \mathrm{Pois}(\lambda t)$
- Expected position of the particle at time $t = \lambda t$
- Particle positions over time is a Poisson process $(N(t))_{t \geq 0}$

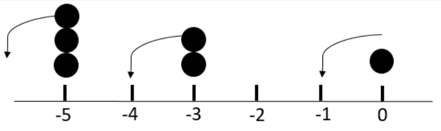


Several particles are positioned at sites of the one-dimensional lattice. Independently, each waits an exponential amount of time and then jumps one site to the left.

- Particle distribution at time 0 is a product of Poisson distributions
- Inter-jump time $\sim Exp(\lambda)$
- Particle distribution at time t is a product of Poisson distributions

Define the configuration $\eta = (\eta(x))_{x\in\mathbb{Z}}$, where $\eta(x)$ is the number of particles at site x.

$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)(f(\eta^{x,-}) - f(\eta)) = \sum_{x\in\mathbb{Z}} \eta(x)(X_x(f))(\eta))$$



$$\eta^{x,-}(y) = \begin{cases} \eta(y) - 1 & \text{if } y = x \\ \eta(y) + 1 & \text{if } y = x - 1 \\ \eta(y) & \text{otherwise} \end{cases}$$

$$X_x(f)(\eta) = f(\eta^{x,-}) - f(\eta)$$

$\{X_x\}_{x\in\mathbb{Z}}$ are operators that define the dynamics

TAZRP: Totally asymmetric zero-range process

A Markov jump process with generator

$$(Lf)(\eta) = \sum_{x \in \mathbb{Z}} g(\eta(x))(X_x(f))(\eta)$$

- totally: jumps occur only to the left
- asymmetric: jumps to the left are more likely to occur than jumps to the right
- zero-range: jump rate depends on the departure occupancy, only
- $g : \{0, 1, 2, \dots\} \to \mathbb{R}_+$ bounded, non-decreasing function with $g(0) = 0$

TAZRP with $g(n) = [n] = \frac{1-q^n}{1-q}$
= q-BOSON of Sasamoto and Wadati[1]
= q-TAZRP of Borodin and Corwin [2]

TAZRP with multispecies (smaller species of particles have priority to hop) [3] [4]

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[1]Sasamoto T, Wadati M, Exact results for one-dimensional totaly asymmetric diffusion models, J. Phys. A 31 (1998) 6057-6071

[2]Borodin A, Corwin I, Macdonald Processes, Probab Theor Rel Fields 158 (2014) 225-400

[3]Kuniba A, Maruyama S, Okado M, Multispecies totally asymmetric zero range process: I. Multiline process and combinatorial R, J Integrable Syst 1 (2016) xyw002

[4]Kuniba A, Maruyama S, Okado M, Multispecies totally asymmetric zero range process: II. Hat relation and tetrahedron equation, J Integrable Syst 1 (2016) xyw008

TAZRP classical: Invariant measures that are translation-invariant and product measures

$$\nu^{\rho}(k) = \frac{1}{Z(\lambda)} \frac{\lambda^{k}}{g(1)\ldots g(k)}$$

- λ is selected so that the expected value of ν^{ρ} is ρ
- $Z(\lambda)$ is a normalizing constant

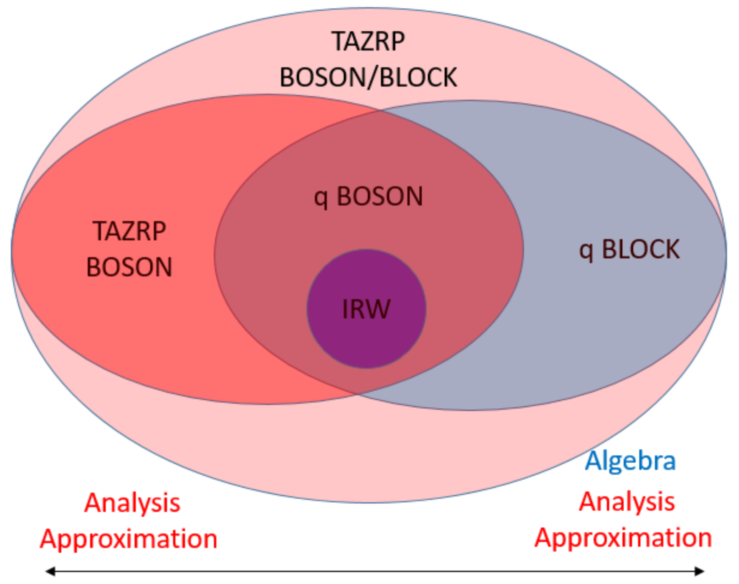
$$\nu^{\rho}(\eta(1), \ldots, \eta(L)) = \nu^{\rho}(\eta(1)) \times \cdots \times \nu^{\rho}(\eta(L))$$

TAZRP multispecies: Invariant measures via matrix product formula

$$P(\eta(1), \ldots, \eta(L)) = Tr(X_{\eta(1)}(\rho_1) \ldots X_{\eta(L)}(\rho_L))$$

$X_k(\rho)$ are operators that satisfy special relationships. Are constructed using R matrix.

MODEL HIERARCHY



Section 2: Algebraic properties

Deformed affine Hecke algebra of type A_{k-1}

Generators

$$T's : T_1, \ldots, T_{k-1} \quad \text{and} \quad X's : X_1, \ldots, X_k, X_1^{-1}, \ldots X_k^{-1}$$

Eigenvalue relations

$$(T_i - 1)(T_i + q) = 0, \quad i = 1, \ldots, k$$

Braid relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \ldots, k$$
$$T_i T_j = T_j T_i, \quad |i - j| > 1$$

Laurent relations

$$X_i X_j = X_j X_i, \quad X_i X_i^{-1} = X_i^{-1} X_i, \quad i, j = 1, \ldots, k$$

Simple action relations

$$T_i X_i T_i = q X_{i+1} \quad i = 1, \ldots, k-1$$
$$T_i X_j = X_j T_i \quad i \neq j, j-1$$

Deformed action relations

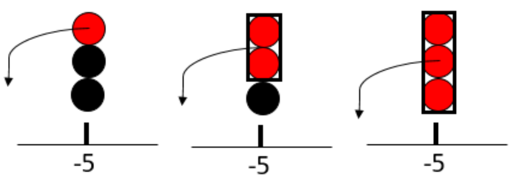
$$X_{i+1} T_i - T_i X_i = T_i X_{i+1} - T_i X_i = (\alpha + \beta X_i)(\gamma + \delta X_{i+1}) \quad i = 1, \ldots, k-1$$
$$T_i X_j = X_j T_i \quad i \neq j, j-1$$

Generators and relations of Hecke algebras

	type A_{k-1}	affine	deformed [5]
Parameters	q	q	$\alpha, \beta, \gamma, \delta$ $q = 1 + \beta\gamma - \alpha\delta$
Generators	$T's$	$T's$, $X's$	$T's$, $X's$
Eigenvalue	✓	✓	✓
Braid	✓	✓	✓
Laurent		✓	✓
Action		simple	deformed

[5] Takeyama Y, A deformation of affine Hecke algebra and integrable stochastic particle systems, J. Phys. A 47(46) (2014)

The model of Takeyama: q-BLOCK



Takeyama's Hamiltonian

$$H(\alpha,\beta,\gamma,\delta) = -\alpha\gamma \sum_{j=1}^{k} \frac{[d_j^+]}{1+\beta\gamma[d_j^+]}$$

$$+\sum_{r=1}^{k} (-\beta\gamma)^{r-1}[r-1]!q^{(r(r-1)/2)}$$

$$\times \sum_{1\le j_1<\cdots<j_r\le k} \frac{q^{d_{j_1}^- +\cdots+d_{j_r}^-}\delta_{j_1,\ldots,j_r}}{\prod_{p=0}^{r-1}(1+\beta\gamma[d_{j_1}^+ + d_{j_1}^- - p])} \mathbf{X_{j_1}\ldots X_{j_r}}$$

When $(\alpha+\beta)(\gamma+\delta)=0$, $H(\alpha,\beta,\gamma,\delta)$ is the generator of a Markov jump process. The dynamics involve movement to the **left** of more than one particle from one site to the neighboring site.

TAZRP: The model of Takeyama (q-BLOCK)

j particles move to the left from a cluster with a particles at

$$\text{rate} = r(a,j) = \frac{s^{j-1}}{[j]} \prod_{p=0}^{j-1} \frac{[a-p]}{1+s[a-1-p]}$$

- for $r=1$, rate $= r(a,1) = \frac{[a]}{1+s[a-1]}$
- for $r=2$, rate $= r(a,2) = \frac{s[a][a-1]}{(1+q)(1+s[a-1])(1+s[a-2])}$

Limiting cases:

- $s=0$, $0<q<1$, q-BOSON of Sasamoto and Wadati[6] [7]
- $s=0$, $q=1$, Independent Random Walks.

[6] Sasamoto T, Wadati M, Exact results for one-dimensional totaly asymmetric diffusion models, J. Phys. A 31 (1998) 6057-6071

[7] van Diejen JF, Emsiz E, Diagonalization of the infinite q-boson system, J Functional Analysis 266 (2014) 5801-5817

Properties of Takeyama's Hamiltonian

Theorem (Takeyama, 2014) Intertwining property

$$H(\alpha,\beta,\gamma,\delta)G_x = G_x(X_1+\cdots+X_n) = G_x\Delta_{1/2}$$

Proposition

Define $Y_i = X_{k+1-i}^{-1}$ for $i = 1, \ldots, k$ and $S_i = T_{k-i}$ for $i = 1, \ldots, k-1$. Then $S's$ and $Y's$ satisfies the all relations of the deformed affine Hecke algebra with parameters $\delta, \gamma, \beta, \alpha$.

$$X_{i+1}^{-1}(X_{i+1}T_i - T_iX_i)X_i^{-1} = T_iX_i^{-1} - X_{i+1}^{-1}T_i = (\delta + \gamma X_{i+1}^{-1})(\beta + \alpha X_i^{-1})$$

$$X_i^{-1}(T_iX_{i+1} - X_iT_i)X_{i+1}^{-1} = X_i^{-1}T_i - T_iX_{i+1}^{-1} = (\delta + \gamma X_{i+1}^{-1})(\beta + \alpha X_i^{-1})$$

Consequence

- $H(\delta, \gamma, \beta, \alpha) G_y = G_y(Y_1 + \cdots + Y_n)$
  $$= G_y(X_n^{-1} + \cdots + X_1^{-1}) = G_y \Delta_{-1/2}$$
- $(H(\alpha, \beta, \gamma, \delta) + H(\delta, \gamma, \beta, \alpha))G$
  $$= G(X_1 + \cdots + X_n + X_1^{-1} + \cdots + X_n^{-1}) = G\Delta$$

$H(\alpha, \beta, \gamma, \delta) + H(\delta, \gamma, \beta, \alpha)$ encodes particle movement to both **left** and **right** (AZRP).

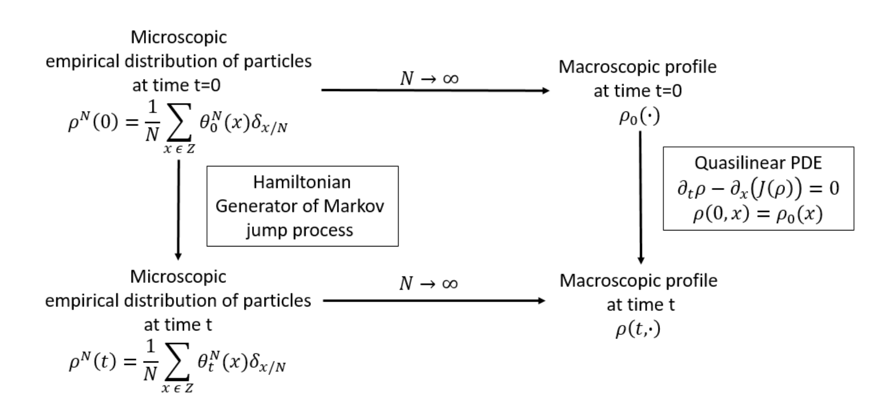
Consequence: Eigenvectors of the Hamiltonian can be constructed from the eigenvectors of the Laplacian. The Laplacian's eigenvectors are calculated via Bethe Ansatz. Knowledge of these eigenvectors help calculate transition probabilities.
For q-BOSON transition probabilities can be calculated [8] [9]

[8] Borodin A, Corwin I, Petrov L, and Sasamoto T, SPECTRAL THEORY for the q-BOSON PARTICLE SYSTEM

[9] Korhonen M and Lee E, The transition probability and the probability for the left-most particle's position of the q-TAZRP

Section 3: Analytic/Asymptotic properties

Hydrodynamic Scaling Limit



TAZRP: Hydrodynamic Scaling Limit

Theorem (Rezakhanlou[10] , 1991)

TAZRP has a hydrodynamic scaling limit given by the solution of the quasi-linear hyperbolic equation of first order

$$\partial_t \rho = \partial_x (J(\rho))$$

$$\rho(0, x) = \rho_0(x)$$

where $J(\rho) = E_{\nu^\rho}[g]$ is the expected microscopic current through a site

[10] Rezakhanlou F, Hydrodynamic Limit for Attractive Particle Systems on Z^d, Commun. Math. Phys. 140, 417-448 (1991)

TAZRP: Hydrodynamic Scaling Limit

Initial particle distribution $\rho(0,x) = \begin{cases} b & x < 0 \\ a & x \geq 0 \end{cases}$

Independent Random Walks

$$\partial_t(\rho) - \partial_x(\rho) = 0$$

$$\rho(t,x) =$$

$$\begin{cases} b & x < -t \\ a & x \geq t \end{cases}$$

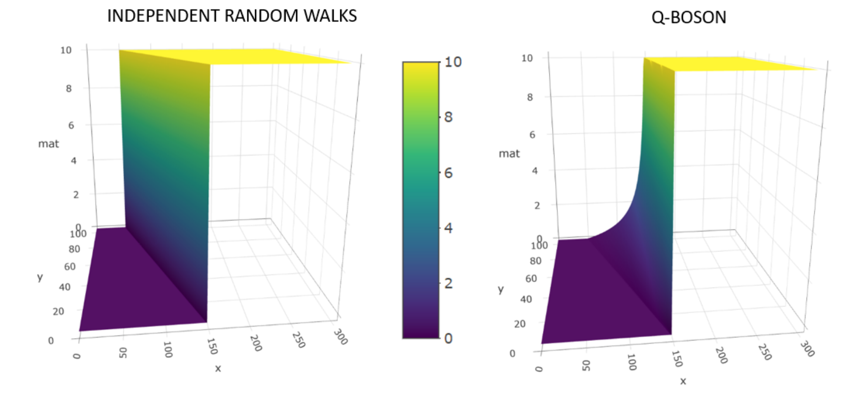
q-Boson

$$\partial_t(\rho) - \frac{1}{(1+(1-q)\rho)^2}\partial_x(\rho) = 0$$

$$\rho(t,x) =$$

$$\begin{cases} b & x < -f(t,b) \\ \frac{1}{1-q}(\sqrt{\frac{t}{-x}} - 1) & -f(t,b) \leq x < -f(t,a) \\ a & x \geq -f(t,a) \end{cases}$$

TAZRP: Hydrodynamic Scaling Limit



Proposition (in progress) (Savu)

For $s \leq q$, the q-BLOCK is attractive. Should the definition of attractiveness be weakened, as in the work of Thierry Gobron?

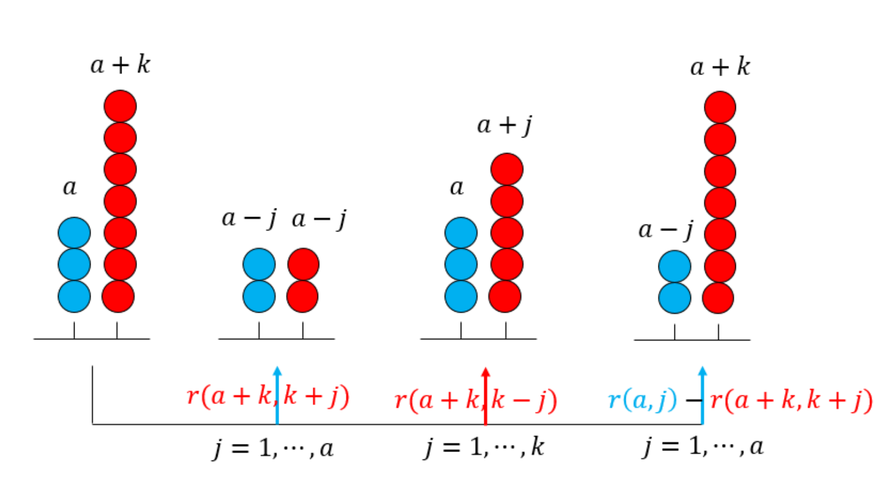
$$s[b+1] = s(1+q+\cdots+q^b) = s(1+q[b]) \leq q + sq[b] = q(1+s[b])$$

$\Rightarrow$

$$r(a+k, j+k) = r(a,j) \times \frac{[j]}{[j+k]} \times \frac{s[a+k]}{1+s[a+k-1]} \times \cdots \times \frac{s[a+1]}{1+s[a]}$$

$$\leq r(a,j) \times \frac{q^k[j]}{[j+k]} \leq r(a,j)$$

The following coupling does not preserve the order of the initial configuration.



Theorem (in progress)

For $s \leq q$, the q-BLOCK has a hydrodynamic scaling limit given by the solution of the quasi-linear hyperbolic equation of first order

$$\partial_t \rho = \partial_x (J(\rho))$$

$$\rho(0, x) = \rho_0(x)$$

where $J(\rho) = E_{\nu^\rho(da)}\Big[\sum_{j=1}^{a} jr(a, j)\Big]$ is the expected microscopic current through a site.

Challenging because it is not clear the kind of attractiveness satisfied by the process.

q-BLOCK: Kolmogorov Forward Equation 2P

$$\frac{dP_t}{dt}(y,x) = P_t(y+1,x) + P_t(y,x+1) - 2P_t(y,x), \quad {\scriptstyle y \geq x+2}$$

$$\frac{dP_t}{dt}(x+1,x) = P_t(x+2,x) + uP_t(x+1,x+1) - 2P_t(x+1,x)$$

$$\frac{dP_t}{dt}(x,x) = P_t(x+1,x) + vP_t(x+1,x+1) - (u+v)P_t(x,x)$$

$$u = r(2,1) = \frac{1+q}{1+s} = (1+q)(1-\lambda)$$

$$v = r(2,2) = \frac{s}{1+s} = \lambda$$

Let P_0 be the solution

$$\frac{dP_t^0}{dt}(y,x) = P_t^0(y+1,x) + P_t^0(y,x+1) - 2P_t^0(y,x), \quad y,x \in \mathbb{Z}$$

satisfying the boundary condition

$$P_t^0(x,x+1) = (u-1)P_t^0(x+1,x) + vP_t^0(x+1,x+1) - (u+v-2)P_t^0(x,x)$$

Then P_t satisfies KF equation for 2 particles

$$P_t(y,x) = \begin{cases} P_t^0(y,x), & \text{for} \quad y \geq x+1 \\ \frac{1}{u}P_t^0(x,x), & \text{for} \quad y = x \end{cases}$$

Weighting $\rightarrow$ RED EQ.
Weighting + Boundary Condition $\rightarrow$ BLUE EQ.

q-BLOCK: Kolmogorov Forward Equation 3P

$$\frac{dP_t}{dt}(z,y,x)=P_t(z+1,y,x)+P_t(z,y+1,x)+P_t(z,y,x+1)-3P_t(z,y,x), \quad z\geq y+2\geq x+4$$

$$\frac{dP_t}{dt}(z,x+1,x)=P_t(z+1,x+1,x)+P_t(z,x+2,x+1)+r_1^2P_t(z,x+1,x+1)-3P_t(z,x+1,x), \quad z\geq x+3$$

$$\frac{dP_t}{dt}(x+1,x,y)=P_t(x+2,x,y)+r_1^2P_t(x+1,x+1,y)+P_t(x+1,x,y+1)-3P_t(x+1,x,y), \quad x\geq y+2$$

$$\frac{dP_t}{dt}(x+1,x,x-1)=P_t(x+2,x,x-1)+r_1^2P_t(x+1,x+1,x-1)+r_1^2P_t(x+1,x,x)-3P_t(x+1,x,x-1)$$

$$\frac{dP_t}{dt}(z,x,x)=P_t(z+1,x,x)+P_t(z,x+1,x)+r_2^2P_t(z,x+1,x+1)-(1+r_1^2+r_2^2)P_t(z,x,x), \quad z\geq x+2$$

$$\frac{dP_t}{dt}(x,x,y)=P_t(x+1,x,y)+r_2^2P_t(x+1,x+1,y)+P_t(x,x,y+1)-(1+r_1^2+r_2^2)P_t(x,x,y), \quad x\geq y+2$$

$$\frac{dP_t}{dt}(x+1,x,x)=P_t(x+2,x,x)+r_1^2P_t(x+1,x+1,x)+r_2^3P_t(x+1,x+1,x+1)-(1+r_1^2+r_2^2)P_t(x+1,x,x)$$

$$\frac{dP_t}{dt}(x,x,x-1)=P_t(x+1,x,x-1)+r_2^2P_t(x+1,x+1,x-1)+r_1^3P_t(x,x,x)-(1+r_1^2+r_2^2)P_t(x,x,x-1)$$

$$\frac{dP_t}{dt}(x,x,x)=P_t(x+1,x,x)+r_2^2P_t(x+1,x+1,x)+r_3^3P_t(x+1,x+1,x+1)-(r_1^3+r_2^3+r_3^3)P_t(x,x,x)$$

Let P_0 be the solution

$$\frac{dP_t^0}{dt}(z,y,x)=P_t^0(z+1,y,x)+P_t^0(z,y+1,x)+P_t^0(z,y,x+1)-3P_t^0(z,y,x), \quad z,y,x\in\mathbb{Z}$$

satisfying the boundary condition

$$P_t^0(x,x+1,y) = (r_1^2-1)P_t^0(x+1,x,y)+r_2^2P_t^0(x+1,x+1,y)+(2-r_1^2+r_2^2)P_t^0(x,x,y) \quad x+1\geq y$$

$$P_t^0(z,x,x+1) = (r_1^2-1)P_t^0(z,x+1,x)+r_2^2P_t^0(z,x+1,x+1)+(2-r_1^2+r_2^2)P_t^0(z,x,x) \quad z\geq x+1$$

Then P_t satisfies KF equation for 3 particles

$$P_t(z,y,x) = \begin{cases} P_t^0(z,y,x), & \text{for } z \geq y+1 \geq x+2 \\ \frac{1}{r(2,1)}P_t^0(z,y,x), & \text{for } z = y \geq x+1 \text{ or } \\ & \quad z-1 \geq y = x \\ \frac{1}{r(2,1)r(3,1)}P_t^0(z,y,x), & \text{for } x = y = z \end{cases}$$

q-BLOCK: Kolmogorov Forward Equation 3p

under the constraints on the transition rates

$$
\begin{aligned}
\textit{Constrain } 1 \quad r(3,1) &= 1 + \frac{r(2,1)(r(2,1)-1)}{1-r(2,2)(2-r(2,1)-r(2,2))} \\
\textit{Constrain } 2 \quad r(3,2) &= r(3,1)r(2,1) \\
\textit{Constrain } 3 \quad r(3,3) &= \frac{r(2,1)r(2,2)^2}{1-r(2,2)(2-r(2,1)-r(2,2))}
\end{aligned}
$$

Weighting → RED EQ.

Weighting + Boundary Condition → BLUE EQ.

Weighting + Boundary Condition + Constrain 2 → GREEN EQ.

Weighting + Boundary Condition + Constrains 2 + Constrains 1, 3 → ORANGE EQ.

Conservative models

MODEL	SEP	ZRP	CRSOS
Total number of particles is conserved	YES	YES	YES
Amenable configuration space	NO	YES	YES
Rates depend on surface gradient	YES	NO	YES

SEP: Simple exclusion process
ZRP: Zero range process
CRSOS: Conservative restricted solid-on-solid model