Almost Elementary C^*-Dynamics and $\mathcal{Z}$-Stability of Crossed Products

Joachim Zacharias

University of Glasgow

Joint work with

Joan Bosa, Francesc Perera and Jianchao Wu

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For a simple separable unital nuclear C*-algebra A tfae:

1. A has finite nuclear dimension;
2. $A \cong A \otimes \mathcal{Z}$, ie A is $\mathcal{Z}$-stable;
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$\mathcal{Z}$-stability minimal regularity condition since $E(A) = E(A \otimes \mathcal{Z})$.

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Definition (nuclear dimension, (Winter, Z))

$dim_{nuc}(A) \leq d$ iff $\forall F \subset\subset A$, $\forall \varepsilon > 0$, $\exists$ finite-dimensional C*-algebra C, cpc maps $\psi : A \to C = C_0 \oplus \ldots \oplus C_d$ and order zero maps ($cpc_\perp$) $\varphi_i : C_i \to A$ such that $\|(\varphi_0 + \ldots + \varphi_d) \circ \psi(a) - a\| < \varepsilon$, $\forall a \in F$.

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Theorem (tracial $\mathcal{Z}$-stability, (Hirshberg-Orovitz))

A simple unital nuclear C-algebra A is $\mathcal{Z}$-stable if $\forall F \subset\subset A$, $\forall \varepsilon > 0$, $\forall n$, $\forall b \in A_+ \setminus \{0\}$, $\exists$ $cpc_\perp$ $\varphi : M_n \to A$ s.t. $\|[F, \varphi((M_n)_{\leq 1})]\| < \varepsilon$ and $1 - \varphi(1) \preceq_{Cu} b$.*

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Very natural class of simple separable unital nuclear C*-algebras come from dynamical systems,

$$\alpha : G \curvearrowright X \qquad \text{or} \qquad \alpha : G \curvearrowright A,$$

with associated crossed product C*-algebras

$$C(X) \rtimes_\alpha G \qquad \text{or} \qquad A \rtimes_\alpha G.$$

Good criteria for their simplicity, tools to determine their K-theory; nuclear if A is and G amenable; UCT automatic for $C(X) \rtimes_\alpha G$ and for many $A \rtimes_\alpha G$.

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Goal:

Identify conditions on $\alpha : G \curvearrowright A$ ensuring classifiability of $A \rtimes_\alpha G$.

For A commutative, ie actions $\alpha : G \curvearrowright X$ on cpt metric spaces, the concept of **almost finite actions** (Kerr) is a very effective tool to prove $\mathcal{Z}$-stablity of $C(X) \rtimes_\alpha G$, close to being equivalent.

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Theorem (Kerr+ generalising Rokhlin dim results by Szabo-Wu-Z)

Crossed products by almost finite actions are $\mathcal{Z}$-stable.
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We define new property, almost elementariness of $\alpha : G \curvearrowright A$, A noncommutative, ensuring classifiability of $A \rtimes_\alpha G$, reducing to $\mathcal{Z}$-stability for $G = \{e\}$, and almost finiteness for $A = C(X)$.

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1. Dynamical strict comparison:

Definition (Kerr (using ideas by Winter))

Given $\alpha : G \curvearrowright X$ we define for open sets $U, V \subseteq X$,

$$U \preceq_G V$$

if $\forall$ compact $K \subseteq U$, $\exists U_1, ..., U_k \subseteq X$ open, $\exists g_1, ..., g_k \in G$ s.t. $K \subseteq U_1 \cup \ldots \cup U_k$ and $g_1 U_1 \ldots g_k U_k$ are disjoint subsets in V.

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For G amenable, $\alpha : G \curvearrowright X$ is **almost finite** if

$\forall F \subset\subset G$, $\forall \varepsilon > 0$, $\exists$ open castle in X s.t.

1. diam(each level) $< \varepsilon$,
2. each shape is (F, ε)-Følner,
3. the remainder of the castle is small: $\exists S_i' \subset S_i, \frac{|S_i'|}{|S_i|} < \varepsilon : X \setminus \bigcup_i S_i B_i \preceq_G \bigcup_i S_i' B_i$.

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A castle can be regarded as a simultaneous approximation of the space and the action **up to a dynamically/tracially small remainder**.

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$$W(A) := (M_\infty(A)_+, \oplus, \precsim_{Cu}); \quad Cu(A) := ((A \otimes K)_+, \oplus, \precsim_{Cu}).$$

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In this case castles are families of orthogonal positive elements covariant along towers.

Definition (Almost Elementariness, covering version, (BPWZ))

$\alpha : G \curvearrowright A$ is AE_{abs} if $\forall F \subset\subset A$, $\forall E \subset\subset G$, $\forall \varepsilon > 0$, $\forall b \in A_+ \setminus \{0\}$,
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Relative almost elementariness AE_{rel} requires existence of tracially small projection $p \in C$ with

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Theorem (BPWZ)

For $A = C(X)$ and G amenable, $\alpha : G \curvearrowright A$ is almost elementary iff $\alpha : G \curvearrowright X$ is almost finite.

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Theorem (BPWZ)

Let A be simple and $\mathcal{Z}$-stable and let $\alpha : G \curvearrowright A$ have the weak tracial Rokhlin property. Then $\alpha : G \curvearrowright A$ is almost elementary.

Thank you very much !