Approximate Representations of Compact Quantum Groups

Jennifer Zhu

University of Waterloo

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Definition

For $\sigma \in M_n(\mathbb{C})^+$ define the seminorm

$$\langle A, B \rangle_\sigma = Tr(B^* A \sigma), \quad \|A\|_\sigma^2 = \langle A, A \rangle_\sigma.$$

Theorem [Gowers-Hatami '16, Vidick '20 (blog post)]

Let G be a finite group, $\epsilon \geq 0$, $\sigma \in M_n(\mathbb{C})^+$ with trace 1, and $f : G \to U_n(\mathbb{C})$ be a map such that

$$\mathbb{E}_{x,y \in G} \| f(x)^* f(y) - f(x^{-1}y) \|_\sigma \leq \epsilon.$$

Then there exists $n' \geq n$, an isometry $V : \mathbb{C}^n \to \mathbb{C}^{n'}$ and a representation $g : G \to U_{n'}(\mathbb{C})$ such that

$$\mathbb{E}_{x,y \in G} \| f(x) - V^* g(x) V \|_\sigma^2 \leq \epsilon.$$

Definition

A CQG is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital C^*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \underset{\min}{\otimes} C(\mathbb{G})$ is a unital, coassociative $*$-homomorphism such that we have the "co-cancellation property":

$$(C(\mathbb{G}) \otimes 1)\Delta(C(\mathbb{G})) = C(\mathbb{G}) \underset{\min}{\otimes} C(\mathbb{G}) = (1 \otimes C(\mathbb{G}))\Delta(C(\mathbb{G})).$$

Definition

A representation of a CQG $\mathbb{G}$ on a finite dimensional Hilbert space $\mathcal{H}$ is an invertible element $U \in B(\mathcal{H}) \otimes C(\mathbb{G})$ such that

$$(id \otimes \Delta)U = U_{12}U_{13}.$$

Translation

Definition [Brannan '23]

For M a von Neumann algebra and CQG $\mathbb{G}$, the M-valued Fourier transform is $\mathcal{F}_M : M\overline{\otimes}L^\infty(\mathbb{G}) \to M\overline{\otimes}\ell^\infty(\widehat{\mathbb{G}})$ given by

$$f \mapsto (1_M \otimes 1 \otimes h)(W_{23}f_{13}) =: \widehat{f},$$

where $W \in M(c_0(\widehat{\mathbb{G}}) \otimes C_r(\mathbb{G}))$ is the right regular representation of $\mathbb{G}$.

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Indeed, considering $W = \bigoplus_{\pi\in \mathrm{Irr}(\mathbb{G})} U^\pi$, with each $U^\pi \in B(\mathcal{H}_\pi) \otimes \mathcal{O}(\mathbb{G})$ being a fixed representative of its class,

$$\widehat{f}(\pi) = (1 \otimes 1 \otimes h)(U^\pi_{23}f_{13}).$$

Proposition [Brannan '23]

The map $\mathcal{F}_M$ has the following properties for $f, g \in M\overline{\otimes}L^\infty(\mathbb{G})$:

1. (Convolution formula) If $k = (1 \otimes h \otimes 1)(f_{12}^*(1 \otimes \Delta)g)$ then $\widehat{k} = \widehat{f}^*\widehat{g}$.
2. (Plancherel formula) We have
$$(1 \otimes \widehat{h}_L)((\widehat{g})^*\widehat{f}) = (1 \otimes h)(g^* f).$$
3. (Inversion formula) For the same k above,
$$k = \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} d(\pi)(1 \otimes Tr_{B(H_\pi)}(\rho_\pi^{-1} \cdot) \otimes 1)((U_{23}^\pi)^* \widehat{k}(\pi)_{12})$$
converges under $\| \cdot \|_{M\overline{\otimes}L^\infty(\mathbb{G})}$.

Further Directions

- Construct a non-local game for which this approximation has meaning.
- Find a CQG that is *not* stable.
- Attempt to translate other classical approximation results.