Fuzzy fermions

Dirac ensembles from fuzzy geometry with a fermion

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Fuzzy geometry

Definition

A (p, q) fuzzy geometry is a spectral triple of KO-dimension $s = q - p$

$$(M_N(\mathbb{C}), V \otimes M_N(\mathbb{C}), D; J, \Gamma)$$

with V a $\mathrm{Cl}_{p,q}$ module, J, Γ are determined by V.

Proposition (Barrett)

$$D(A) = \sum_{I \subseteq \{1,\ldots,p+q\}} \gamma^I \otimes (K_I A \pm A K_I)$$

for $K_I \in M_N(\mathbb{C})$, $K_I^* = \pm K_I$.

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(0, 1) Fuzzy Dirac ensemble

Example

In signature $(0, 1)$ the fuzzy geometries are

$$(M_N(\mathbb{C}), M_N(\mathbb{C}), D = [H, \cdot]; J = \cdot^*)$$

for $H \in M_N(\mathbb{C})$ self-adjoint.

Dirac ensemble

Definition

A fuzzy geometry Dirac ensemble is a choice of (p, q) together with a probability distribution $P(D) = \frac{1}{Z}e^{-S(D)}$ on the space of Dirac operators $\mathcal{D}$.

Random matrix theory

The corresponding matrix model is given by

$$Z_N = \int_{\mathcal{H}_N^k} e^{-S(D(\{K_I\}))} d^k K_I$$

where $D(\{K_I\})$ is the Dirac operator

$$D(A) = \sum_{I \subseteq \{1,\dots,p+q\}} \gamma^I \otimes i^{\epsilon_I}(K_I A \pm A K_I)$$

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$(0, 1)$ Fuzzy Dirac ensemble

Example

In signature $(0, 1)$, the fuzzy geometry is

$$(M_N(\mathbb{C}), M_N(\mathbb{C}), D = [H, \cdot]; J = \cdot^*)$$

so the associated matrix model for $S(D) = \mathrm{Tr}(D^2)$ is

$$Z_N = \int_{\mathcal{D}} e^{-\mathrm{Tr}(D^2)} dD = \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(H^2)+2\,\mathrm{Tr}(H)^2} dH.$$

This is a unitarily invariant one-matrix two-trace model.

Fermionic action

Spectral action

The action in the NCG standard model has two parts, the gauge/metric part and the fermionic part

$$S(D,\psi) = \mathrm{Tr}(f(D)) + \langle\psi, D\psi\rangle$$

Dirac observables

For $O : \mathcal{D} \to \mathbb{R}$ an observable

$$\begin{aligned}\langle O\rangle &= \frac{1}{Z_N}\int_{\mathcal{D},M_N(\mathbb{C})} O(D)e^{-S_g(D)-\langle\psi,D\psi\rangle}\,d\psi dD,\\ &= \frac{1}{Z_N}\int_{\mathcal{D}} O(D)e^{-S_g(D)}\int_{M_N(\mathbb{C})} e^{-\langle\psi,D\psi\rangle}\,d\psi dD.\end{aligned}$$

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Fermionic action

Fermionic (Grassman, Berezin) integral

$$\psi = \sum_{(i,j)\in I} \psi_{i,j} E_{i,j}, \quad \psi_{i,j}\psi_{k,l} = -\psi_{k,l}\psi_{i,j}$$

so

$$f(\psi) = \sum_{J\subset I} f_J \prod_{(a,b)\in J} \psi_{a,b}$$

Then

$$\int_{Ber} f(\psi) d\psi := f_I$$

Proposition

$$\int_{Ber} e^{-\langle \psi, D\psi\rangle} d\psi \propto \det(D), \quad \int_{Ber} e^{-\langle J\psi, D\psi\rangle} d\psi \propto \mathrm{Pf}(\langle J\cdot, D\cdot\rangle)$$

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Fermionic action for the $(0,1)$ fuzzy geometry

Example

If $D = [H, \cdot]$, $\mathrm{spec}(H) = \{\lambda_i\}_{i=1}^N$, then $\mathrm{spec}(D) = \{\lambda_i - \lambda_j\}_{i,j=1}^N$.
So $\det(D) = 0^N \prod_{i \neq j}(\lambda_i - \lambda_j) = 0$.

There is hope

$$P(D) = \frac{1}{Z} e^{-S_g(D)} \det(D) = \frac{e^{-S_g(D)} \det(D)}{\int_{\mathcal{D}} e^{-S_g(D)} \det(D) dD}$$

Idea: $D \rightsquigarrow D + m$.

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Adding in a finite space

Almost fuzzy spectral triples

Consider

$$(M_N(\mathbb{C}), V \otimes M_N(C), D) \otimes (A_F, H_F, D_F)\,.$$

This yields Yang-Mills and Higgs like effects.

The product triple

$$(M_N(\mathbb{C}) \otimes A, V \otimes M_N(\mathbb{C}) \otimes H_F, D \otimes 1 + 1 \otimes D_F)$$

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The $(0,1)$ fuzzy fermion model

Ingredients

The $(0,1)$ fuzzy geometry

$$(M_N(\mathbb{C}), M_N(\mathbb{C}), [H,\cdot]; \cdot^*)$$

and fermion space

$$(\mathbb{C}, \mathbb{C}, m; \overline{\cdot})\,.$$

Our almost fuzzy model

The $(0,1)$ fuzzy geometry with a single mass m fermion model is

$$\left(M_N(\mathbb{C}), M_N(\mathbb{C})\otimes\mathbb{C}^2, \begin{pmatrix} 0 & [H,\cdot]-im \\ [H,\cdot]+im & 0 \end{pmatrix}; \begin{pmatrix} \cdot^* & 0 \\ 0 & \cdot^* \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

The partition function

$$
\begin{aligned}
Z_N &= \int_{\mathcal{D}} \int_{M_N(\mathbb{C})\otimes\mathbb{C}^2} e^{-\operatorname{Tr}(D^2)-\langle J\psi, D\psi\rangle} d\psi dD, \\
&= \int_{\mathcal{H}_N} e^{-\left[4N\operatorname{Tr}(H^2)-4\operatorname{Tr}(H)^2+2m^2\right]} \operatorname{Pf}(\langle J\cdot, D\cdot\rangle) dH, \\
&\propto \int_{\mathcal{H}_N} e^{-4N\operatorname{Tr}(H^2)+4\operatorname{Tr}(H)^2} \prod_{i<j}(m^2+(\lambda_i-\lambda_j)^2) dH, \\
&\propto \int_{\mathbb{R}^N} e^{-4N\sum\lambda_i^2+4(\sum\lambda_i)^2} \prod_{i<j}(m^2+(\lambda_i-\lambda_j)^2) \prod_{i<j}(\lambda_i-\lambda_j)^2 d^N\lambda
\end{aligned}
$$

Finding the spectral density

Goal

Find ρ such that for $O : \mathbb{R} \to \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\sum_i O(\lambda_i)\right) = \int_{\mathbb{R}} O(x)\rho(x)dx.$$

This requires finding the spectral density of a single-matrix two-trace random matrix model.

Some analysis later

We get:

- For any compact, symmetric set $\Sigma = \cup_{i=1}^{r}[a_i, b_i]$ a function ρ_Σ.
- Several conditions in terms of the action and ρ_Σ.

For exactly one choice of $\{a_i, b_i\}$ these conditions are satisfied and then ρ_Σ is the spectral density.

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This ρ and results

The equation for ρ

$$\rho(x) = \frac{2}{\beta} s_\Sigma(x) p_{\Sigma,S_g}(x) + \frac{2}{\beta} s_\Sigma(x) \int_\Sigma K_{\Sigma,m}(x,y)\rho(y)dy$$

and conditions are, e.g.,

$$\int_\Sigma \rho(x)dx = 1.$$

Some interesting limits

- If $m \to \infty$, $K \to 0$.
- If $m \to 0$, $K(x,y) \to -\frac{1}{s_\Sigma(x)}\delta(y-x)$.

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Plots: Gaussian action, $S(D) = \mathrm{Tr}(D^2)$

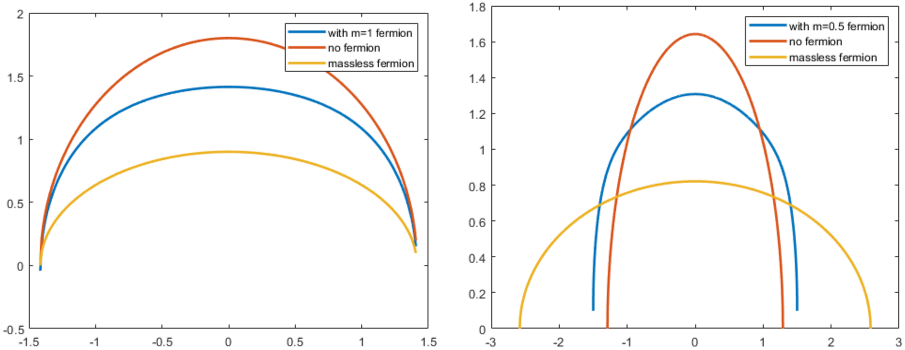

Figure: On the left, ρ for various cases on the same support. On the right on different supports to equalize the total integral. Orange line: no fermion. Blue line: massive fermion. Yellow line: massless fermion.

Plots: Quartic action, $S(D) = g_4 \operatorname{Tr}(D^4) + g_2 \operatorname{Tr}(D^2)$

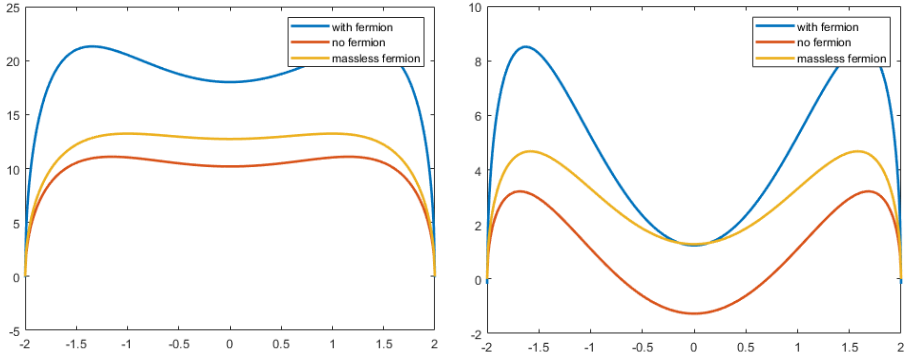

Figure: ρ for the quartic action for two different choices of coupling constants, on the left in the one-cut case, on the right near the phase transition. There is a secondary effect here, as the mass affects the coupling constants. Orange line: no fermion. Blue line: massive fermion. Yellow line: massless fermion.

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