An Invitation to Random Noncommutative Geometries

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Motivation: Quantum Gravity

- When constructing a theory of Quantum Gravity one tries to make sense of a partition function that "sums or integrates over manifolds" and a path integral over some matter field:

$$Z = \sum \int e^{-S(g,X)} dgdX.$$

- What is *dg*? Usually this interpreted as summing over geometric degrees of freedom.
- We would like to consider such integrals in the setting of Noncommutative Geometry (NCG).

Motivation: Noncommutative Geometry

- In NCG, Spectral Triples $(\mathcal{A}, \mathcal{H}, D)$ mimic the data given by a smooth Riemannian manifold with spin structure.
- $\mathcal{A}$ is a involutive complex algebra acting by bounded operators on a Hilbert space $\mathcal{H}$, and D is a self-adjoint (in general unbounded) operator acting on $\mathcal{H}$. This data is required to satisfy some regularity conditions.[1]
- In particular Riemmanian spin^c manifolds can be reconstructed from their spectral triples via Conne's Reconstruction theorem.[2]

[1] Alain Connes. "Noncommutative geometry and reality". In: *Journal of Mathematical Physics* 36.11 (1995), pp. 6194–6231.
[2] Alain Connes. "On the spectral characterization of manifolds". In: *Journal of Noncommutative Geometry* 7.1 (2013), pp. 1–82.

Fuzzy Spectral Triples

- In 2015 John Barrett[3] proposed parameterizing the partition function of a theory of quantum gravity by the moduli space of all possible Dirac operators for a fixed algebra and Hilbert space:

$$\int_g e^{-S(g)} dg \to \int_{\mathcal{D}} e^{-\operatorname{Tr} S(D)} dD.$$

- To make sense of these integrals he considered certain natural real finite spectral triples where the algebra and Hilbert space are replaced by an algebra of matrices. Such spectral triples $(M_N(\mathbb{C}), M_N(\mathbb{C}) \otimes V, D)$ are called fuzzy geometries.

- These integrals are in fact matrix integrals.[4]

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[3] John W Barrett. "Matrix geometries and fuzzy spaces as finite spectral triples". In: *Journal of Mathematical Physics* 56.8 (2015), p. 082301.

[4] John W Barrett and Lisa Glaser. "Monte Carlo simulations of random non-commutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 49.24 (2016), p. 245001.

Fuzzy spectral triples

- The Dirac operators of fuzzy geometries can be written in term of gamma matrices and the commutators or anti-commutators with Hermitian matrices H and skew-Hermitian matrices L. For example:
- $D = \{H, \cdot\}$
- $D = -i[L, \cdot]$
- $$\gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

  Then,

  $$D = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot],$$

  where L_1, L_2 are both skew-Hermitian.
- The entries of these matrices are not specified in Barrett's classification. They are our geometric degrees of freedom!

Dirac Ensembles

- The general form of these Dirac operators of a fuzzy geometry is

$$D = \sum_j \alpha_i \otimes [L_j, \cdot] + \sum_k \beta_k \otimes \{H_k, \cdot\}$$
$$= \sum_\ell \alpha_\ell \otimes \{L_\ell, \cdot\} + \sum_r \beta'_\ell \otimes [H_r, \cdot]$$

where the alpha's and beta's are certain products of gamma matrices that depend on the space of spinors. H's are Hermitian and L skew-Hermitian $N \times N$ matrices.

Example

- We refer to a fuzzy geometry $(M_N(\mathbb{C}), M_N(\mathbb{C}) \otimes V, D)$ equipped with a probability distribution on the entries of D as a *Dirac ensemble*.

- For example let our spectral triple be $(M_N(\mathbb{C}), M_N(\mathbb{C}) \otimes \mathbb{C}, \{H, \cdot\})$ with
$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD$$
where g is a coupling constant.

- The measure becomes the Lebesgue measure on the space of $N \times N$ Hermitian matrices:
$$dD = dH = \prod_{i=1}^{N} dH_{ii} \prod_{1 \leq i < j \leq N} d(\operatorname{Re}(H_{ij}))\, d(\operatorname{Im}(H_{ij})).$$

A type $(1,0)$ ensemble

- The integral

$$Z = \int_{\mathcal{D}} e^{-g\,\mathrm{Tr}\,D^2 - \mathrm{Tr}\,D^4}\, dD$$

then becomes a bi-tracial matrix integral

$$= \int_{\mathcal{H}_n} \exp(-(2Ng\mathrm{Tr}H^2 + 2g(\mathrm{Tr}H)^2 + 2N\mathrm{Tr}(H^4)$$
$$+ 8\mathrm{Tr}H\mathrm{Tr}H^3 + 6(\mathrm{Tr}H^2)^2))dH.$$

- In general Dirac ensembles with polynomial potentials are bi-tracial multi-matrix ensembles.

A quartic type $(2,0)$ ensemble

- The integral

$$Z = \int_{\mathcal{D}} e^{-g \operatorname{Tr} D^2 - \operatorname{Tr} D^4} dD,$$

where

$$\begin{aligned}
\operatorname{Tr} D^2 &= 4N \left(\operatorname{Tr} H_1^2 + \operatorname{Tr} H_2^2\right) + 4\left(\left(\operatorname{Tr} H_1\right)^2 + \left(\operatorname{Tr} H_2\right)^2\right) \\
\operatorname{Tr} D^4 &= 4N \left(\operatorname{Tr} H_1^4 + \operatorname{Tr} H_2^4 + 4 \operatorname{Tr} H_1^2 H_2^2 - 2 \operatorname{Tr} H_1 H_2 H_1 H_2\right) \\
&+ 16 \left(\operatorname{Tr} H_1 \left(\operatorname{Tr} H_1^3 + \operatorname{Tr} H_2^2 H_1\right)\right. \\
&+ \left.\operatorname{Tr} H_2 \left(\operatorname{Tr} H_1^2 H_2 + \operatorname{Tr} H_2^3\right) + \left(\operatorname{Tr} H_1 H_2\right)^2\right) \\
&+ 12 \left(\left(\operatorname{Tr} H_1^2\right)^2 + \left(\operatorname{Tr} H_2^2\right)^2\right) + 8 \operatorname{Tr} H_1^2 \operatorname{Tr} H_2^2.
\end{aligned}$$

Dirac ensembles

- Adding matter fields to this framework is the result of more recent work. See Luuk's talk today!
- There has also been work aimed at incorporating the Standard Model.[5]
- Ideally, since we have considered a class of finite dimensional spectral triples, we would like to find a continuum limit as the matrix size N goes to infinity and relate these models to physics.
- Additionally, as mathematical objects they are inherently interesting, especially from the perspective of Random Matrix Theory (RMT).

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[5] Carlos I Perez-Sanchez. "On multimatrix models motivated by random noncommutative geometry II: A Yang-Mills-Higgs matrix model". In: *Annales Henri Poincaré*. Vol. 23. 6. Springer. 2022, pp. 1979–2023.

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The distributions of eigenvalues

- In RMT one studies the *bounded* distribution of eigenvalues of random matrices in the large N limit.
- In spectral geometry one uses the spectra of an operator, such as the Dirac operator or Laplacian, to recover geometric information using a heat kernel expansion. However, this is done using asymptotic properties of the *unbounded* spectrum.
- Question: how do we recover/interpret geometric properties of Dirac Ensembles?
- Proposed Answer: we study spectral phase transitions!

Spectral phase transitions

- A *spectral phase transition* is when the number of connected components of the support of the eigenvalue distribution of a random matrix changes.
- The most common example is when the eigenvalue distribution of a random matrix is supported on a single interval and then for some value of the coupling constants the support splits into two intervals.[6]

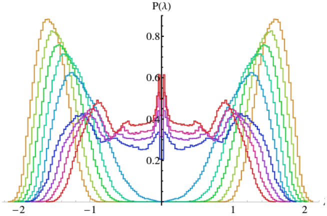

Figure: The eigenvalues of $S(D) = Tr(gD^2 + D^4)$ for $N = 10$ and g =1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5. The lines are coloured from red through to yellow.

[6] John W Barrett and Lisa Glaser. "Monte Carlo simulations of random non-commutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 49.24 (2016), p. 245001.

Spectral phase transitions

- Qualitatively, Barrett and Glaser found that near the spectral phase transition the spectrum of many Dirac ensembles strongly resembled that of the fuzzy sphere. Later a metric between spectra was defined. The spectral distance between many models and the fuzzy sphere was zero near their phase transition.

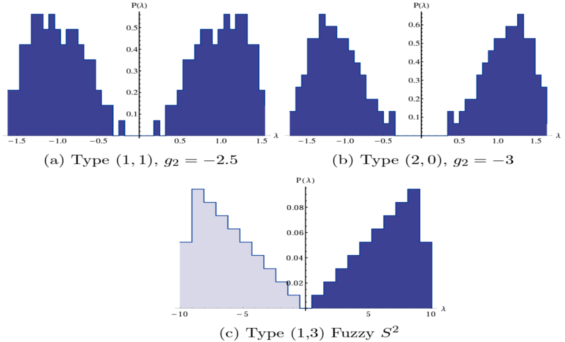

Figure: The eigenvalue distributions near the phase transition compared with the fuzzy sphere for N=10.

Spectral phase transitions

- For simple Dirac ensembles the phase transition can be found explicitly[7] but in general requires numerical methods.[8]
- Additionally, there are notations of spectral dimension and volume[9] as well as an algorithm for generating states of the random fuzzy geometry.[10]
- Further analytical results are needed.

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[7] Masoud Khalkhali and Nathan Pagliaroli. "Phase transition in random noncommutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 54.3 (2020), p. 035202.

[8] Lisa Glaser. "Scaling behaviour in random non-commutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 50.27 (2017), p. 275201, Hamed Hessam, Masoud Khalkhali, and Nathan Pagliaroli. "Bootstrapping Dirac ensembles". In: *Journal of Physics A: Mathematical and Theoretical* 55.33 (2022), p. 335204.

[9] John W Barrett, Paul Druce, and Lisa Glaser. "Spectral estimators for finite non-commutative geometries". In: *Journal of Physics A: Mathematical and Theoretical* 52.27 (2019), p. 275203.

[10] L Glaser. "Computational explorations of a deformed fuzzy sphere". In: *arXiv preprint arXiv:2304.13002* (2023).

1-Cut Case

- When we numerically graph this distribution we find that at some critical value of g it dips below zero and splits into a two-cut case:

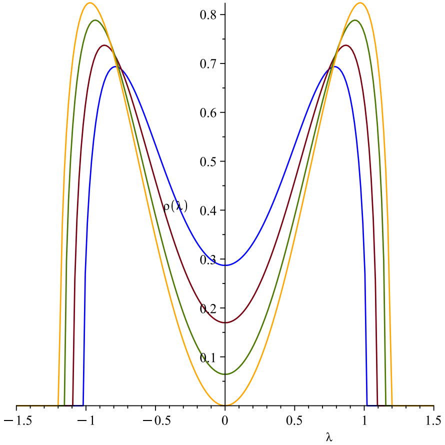

Figure: The equilibrium measure for $(1,0)$ from the single cut analysis.

1-Cut Case

- The spectral density function of H in the simple example is of the form

$$\Psi(x) = \frac{1}{\pi}(-4a^2 + \frac{1}{2a^2} + 4x^2)\sqrt{4a^2 - x^2}_+.$$

- Where $\text{supp}\Psi = [-2a, 2a]$ and a is found as the solution of

$$0 = 192a^8 48a^4 - 4ga^2 - 1$$

for a given value of g.

The Phase Transition

- A precise critical value is found by setting $\Psi(x) = 0$ and $x = 0$ and isolating for a, giving us $a_c = \frac{1}{\sqrt[4]{8}}$.
- Plugging a_c into the above polynomial we find

$$g_c = -4\sqrt{2}.$$

,

This matches Monte Carlo simulations!

2-cut case

- The spectral density function for H in this case is of the form

$$\Psi(x) = \frac{2}{\pi}|x|\sqrt{(x^2 - a^2)(b^2 - x^2)}_+,$$

- Where the supp$\Psi = [-b, -a] \cup [a, b]$ and

$$a^2 = -\frac{1}{8}g + \frac{\sqrt{2}}{2}, \tag{1}$$

$$b^2 = -\frac{1}{8}g - \frac{\sqrt{2}}{2}. \tag{2}$$

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2D quantum gravity from Dirac ensembles

- Random matrix theory has been known to have connections to 2D gravity:
    - The Kontsevich model and Witten's conjecture.
    - Liouville quantum gravity (LQG) i.e. 2D conformal field theories coupled to gravity.
    - More recently Jackiw-Teitelboim (JT) gravity.
- Of particular interest is LQG. Physicists in the late 80's and 90's knew heuristically that asymptotics of random matrix models contained artifacts of LQG.

The double scaling limit

Rough idea:

- The Feynman diagrams associated with matrix integrals are surfaces with embedded graphs (called maps) that can be thought of as discretized Riemann surfaces.
- If the coupling constants of the models were tweaked such that the number of polygons that form maps goes to infinity, one would in essence be counting Riemannian surfaces.
- These critical points exist in many models, in particular we have recently shown that they exist in some Dirac ensembles! In particular we are able to show these models have the same critical exponents and partition functions as models 2D conformal field theory coupled to gravity.[11]

[11] Hamed Hessam, Masoud Khalkhali, and Nathan Joseph Pagliaroli. "Double scaling limits of Dirac ensembles and Liouville quantum gravity". In: *Journal of Physics A: Mathematical and Theoretical* (2022).

The double scaling limit

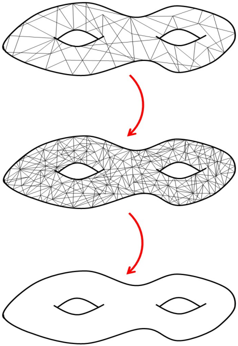

Figure: Intuitively if one fine tunes coupling constants of matrix models such that the number of polygons in maps goes to infinity, maps are replaced by smooth surfaces.

Liouville quantum gravity

- Consider now the quartic type $(1,0)$ Dirac ensemble from earlier

$$Z = \int_{\mathcal{D}} e^{-t_2 \operatorname{Tr} D^2 - t_4 \operatorname{Tr} D^4} dD$$

then becomes a bi-tracial matrix integral

$$= \int_{\mathcal{H}_n} \exp(-(2Nt_2 \mathrm{Tr} H^2 + 2t_2(\mathrm{Tr} H)^2 + 2t_4 N \mathrm{Tr}(H^4) + 8t_4 \mathrm{Tr} H \mathrm{Tr} H^3 + 6t_4(\mathrm{Tr} H^2)^2)) dH.$$

- One can show that near critical points the F_g's have an asymptotic expansion of the form:

$$\operatorname{sing}(F_g) = C_g \, (t_4 - t_c)^{5(1-g)/2}$$

except when $g = 1$,

$$\operatorname{sing}(F_1) = C_1 \, \log(t_4 - t_c).$$

Liouville quantum gravity

- We can define a new formal series

$$u(y) = \sum_{g=0}^{\infty} \text{sing}(F_g) y^{5(1-g)/2},$$

then $u''(y)$ satisfies the Painlevé I equation to all orders

$$y = (u''(y))^2 - \frac{1}{3} u^{(4)}(y).$$

- The Liouville minimal model of conformal field theory coupled to gravity predicts that its "generating function of surfaces" should satisfy this equation!

Liouville quantum gravity

- Different matrix models are associated to different so called minimal models whose F_g's satisfy their own differential equation.
- One can find such models by examining how the spectral density function scales near the critical point(s).

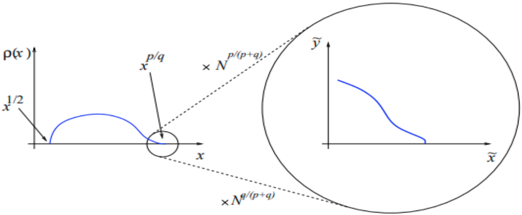

Figure: Borrowed from "Universal scaling limits of matrix models, and (p, q) Liouville gravity" by M. Bergère and B. Eynard.

The phases of the quartic type $(1, 0)$ Dirac ensemble

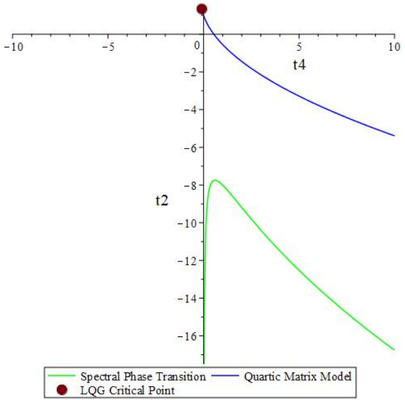

Figure: The phase diagram of the quartic Dirac ensemble.

The phases of the quartic type $(1,0)$ Dirac ensemble

- Curve of the spectral phase transition:

$$t_2 = -\frac{5t_4 + 3}{\sqrt{t4}}.$$

- The quartic Hermitian matrix model's curve:

$$t_2 = -\frac{(1+12t_4)^{3/2} - 4 - 144t_4 + (36t4 + 3)\sqrt{1+12t_4}}{72t_4}.$$

Liouville quantum gravity summary

- These minimal models and critical exponents correspond to representations of the conformal group in two dimensions classified by two integers (p, q) with critical exponents p/q. For example:
  - $(3, 2)$ is called pure gravity and corresponds to the cubic and quartic type $(1, 0)$ Dirac ensembles
  - $(5, 2)$ is called Lee-Yang edge singularity and corresponds to the hexic type $(1, 0)$ Dirac ensemble.
- In general single trace single matrix model correspond to type $(p, 2)$ minimal models and general (p, q) can be found in multi-matrix models. What about other Dirac ensembles?

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A brief summary

- Dirac ensembles are path integrals over fuzzy geometries that can be realized as matrix integrals.
- In the spectral approach, one can interpret geometric features, such as volume and dimension.
- In the graphical approach, certain models one can be connected to LQG.
- For more information please see our recent review article.[12]

[12] Hamed Hessam et al. "From noncommutative geometry to random matrix theory". In: *Journal of Physics A: Mathematical and Theoretical* 55.41 (2022), p. 413002.

Open problems

- Further work is needed to bring the standard model and fermions into the picture.
- Investigate the limiting eigenvalue distribution and critical points of Dirac ensembles with more complicated potentials.
- Are there conformal field theories associated with higher dimensional Dirac ensembles?
- What other geometric data can one extract from Dirac ensembles?
- Is there a way to interpret the Feynman diagrams for complicated Dirac ensembles as higher dimensional discrete spaces? This is done when studying random tensor integrals.

Thank you for listening!
Questions?