Discrete inclusions of C*-algebras

Roberto Hernández Palomares (robertohp.math@gmail.com)
joint with Brent Nelson
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UNIVERSITY OF WATERLOO

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Overview

Main theme:

$\diamondsuit$ Transfer subfactor techniques to C*-algebras.

Goals:

- $\spadesuit$ Obtain class of C*-inclusions admitting standard invariant,
- $\heartsuit$ Characterize C*-discreteness explicitly,
- $\clubsuit$ Applications and examples.

vN Algs & subfactors chronology

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▶ [Müg03] $\left({}_N L^2 M_N =: Q \in \mathcal{C} = \langle {}_N L^2 M_N,\ \oplus,\ \overline{\cdot},\ \boxtimes_N \rangle\right)$
$\rightsquigarrow$ Classification small index subfactors $N \subset M$.

Unitary tensor categories in Nature

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$\therefore$ $\mathcal{C}$ acts on operator algebra A via generalized fiber functor:

$$F : \mathcal{C} \overset{\otimes}{\to} \mathsf{Bim}_{\mathsf{fgp}}(A).$$

Discrete subfactors & quantum dynamics

Discrete subfactors: [ILP98]

$$\underbrace{(N,\tau)\overset{E}{\subset} M}_{\text{discrete}} \quad\Leftrightarrow\quad {}_N L^2(M)_N \;\cong \bigoplus_{K\in\mathrm{Irr}(\mathrm{Bim}_{\mathrm{fgp}}(N))} K^{\oplus n_K},\;\; n_K\in\mathbb{N}\cup\{0\}$$

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Theorem ([JP19] Largest class admitting standard invariant)

$$\left\{ (N,\tau) \overset{E}{\subset} M \;\middle|\; \begin{array}{l} \textit{discrete}, \\ N' \cap M \cong \mathbb{C} \end{array} \right\} \leftrightarrow \left\{ \mathcal{C} \overset{F}{\curvearrowright} N,\; \mathbb{M} \in W^*\mathrm{Alg}(\mathcal{C}) \right\}$$

$\therefore$ *Discrete subfactors determined by a quantum dynamics:*

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Construction/classification discrete subfactors

- Construct/classify quantum dynamics: $\mathcal{C} \curvearrowright N$,
- ●● Describe $W^*\mathrm{Alg}(\mathcal{C})$. (Internal to $\mathcal{C}$)

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$$\forall b \in B \quad \sum u_i E(v_i b) = b = \sum E(b u_i) v_i.$$

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Infinite index C*-subalgebras?

$$\left(A \overset{E}{\subset} B\right) \leftrightarrow \left({}_B B_B, \underbrace{{}_A\mathcal{B}_A,}_{\in \mathrm{C}^*\mathrm{Alg}(\mathcal{C})} \underbrace{\mathcal{C} \overset{F}{\curvearrowright} A}_{\text{outer}}\right).$$

Infinite index inclusions in practice

- The canonical B-A correspondence ${}_B\mathcal{B}_A$:

$$\underbrace{A \overset{E}{\subset} B}_{\text{faithful}} \rightsquigarrow \langle b_1 | b_2 \rangle_A := E(b_1^* b_2) \rightsquigarrow \mathcal{B} = \overline{B\Omega}^{\|\cdot\|_A},$$

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Example (Reduced crossed products)

Given action $F : \mathcal{C} \to \mathrm{Bim_{fgp}}(A)$ and $\mathcal{C}$-graded C*-algebra:

$$A \rtimes_{F,r} \mathbb{B} = \mathrm{C}^*_r\left(\bigoplus_{c\in\mathrm{Irr}(\mathcal{C})} F(c) \otimes \mathbb{B}(c)\right)$$

Synthetic examples

Reduced crossed products by outer group actions are C*-discrete:

$$\Gamma \overset{\alpha}{\curvearrowright} A \rightsquigarrow \left\{ A \overset{E}{\subset} A \rtimes_{\alpha,r} \Lambda \right\}_{\Lambda \leq \Gamma} \subset \mathrm{C}^*\mathsf{Disc}.$$

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- *Genuine* W*/C*-algebra objects are abundant:

•• $\mathbb{G}$: CQG $\overset{\text{T-K}}{\rightsquigarrow}$ $\mathrm{Rep}_f(\mathbb{G}) \curvearrowright \mathrm{Hilb}_f$ $\overset{\text{[JP17]}}{\rightsquigarrow}$ $\mathbb{C}[\mathbb{G}]$,

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The C*-discrete family

$$\underbrace{\{A \rtimes_{\alpha,r} \Gamma\}}_{\text{discrete groups}} \subset \underbrace{\{\mathsf{Ind}_W < \infty\}}_{\text{Q-systems}} \subset \underbrace{\{A \rtimes_{F,r} \mathbb{B}\}}_{C^*\text{-alg objects}} \subseteq \mathrm{C}^*\mathsf{Disc}$$

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Theorem ([**HPN23**])

$$\left\{ A \overset{E}{\subset} B \,\middle|\, C^*\text{-}\mathit{discrete}\ A' \cap B \cong \mathbb{C} \right\} \leftrightarrow \left\{ \mathcal{C} \overset{F}{\curvearrowright} A,\ \mathbb{B} \in C^*\mathrm{Alg}(\mathcal{C}) \right\}$$

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sketch:

$(\Leftarrow)$: take reduced crossed product $A \overset{E'}{\subset} A \rtimes_{F,r} \mathbb{B}$.
$(\Rightarrow)$: $F : \mathcal{C}_{A \subset B} \hookrightarrow \mathrm{Bim_{fgp}}(A)$,

$$B^{\diamondsuit} \cong \underbrace{\bigoplus_{K \in \mathrm{Irr}(\mathrm{Bim_{fgp}}(A))} K \otimes \mathrm{Hom}_{A-A}(K \to B\Omega) \subset B}_{*\text{-subalgebra!}}.$$

Eg: Cores of Cuntz algebras

Example

$$n \in \mathbb{N},\ \underbrace{\mathbb{T} \curvearrowright \mathcal{O}_n}_{\text{gauge}} \rightsquigarrow \mathcal{O}_n^{\mathbb{T}} \cong \mathrm{UHF}_{n^\infty} =: A$$

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Sketch:

$$\underbrace{b \overset{\lambda_1}{\mapsto} s_1 b s_1^*}_{\text{one-sided Bernoulli}} \rightsquigarrow F_1 : \mathrm{Hilb}(\mathbb{Z}) \to \mathrm{Bim_{fgp}}(A),$$

$$k \mapsto \left(s_1^k (s_1^*)^k\right)_{\lambda_1^k}[A]_A.$$

$$\mathcal{O}_n^{\diamondsuit} = * - \mathrm{Alg}(\{s_i\}_1^n) \overset{\text{dense}}{\subset} \mathcal{O}_n.$$

$$\therefore \mathcal{O}_n \cong \mathrm{UHF}_{n^\infty} \rtimes_{F_1, r} \mathbb{C}[\mathbb{Z}].$$

Eg: A-valued semicircular systems

Example

$$\left\{\eta_{ij} : A \overset{\text{c.p.}}{\to} A\right\}_{i,j\in I} \rightsquigarrow \underbrace{\eta = \{\eta_{ij}\} : A \to A \otimes \mathrm{End}(\ell^2(I))}_{\text{covariance matrix}}.$$

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- $\exists\tau$ faithful compatible trace: $\tau(\eta_{ij}(x)y) = \tau(x\eta_{ij}(y))$ and $A \otimes_{\eta_{i,j}} A \in \mathrm{Bim}_{\mathrm{fgp}}(A)$ $\Rightarrow \underbrace{A \overset{E}{\subset} \hat{\Phi}(A,\eta)}_{\text{C*-discrete}} := \mathrm{C}^*\left(A \cup \{\ell(\xi_i) + \ell(\xi_i)^*\}_i\right) \subset \mathrm{End}^\dagger(\mathcal{F}(\eta)_A)$.

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- Assuming $\#I \in \mathbb{N}$:
  $A \subset \hat{\Phi}(A,\eta)$ irreducible $\Leftrightarrow \mathcal{F}(\eta)^A \cong \mathbb{C}$.

$\therefore$ $\hat{\Phi}(A,\eta)$ is as a crossed product!

C*-discrete extensions & simplicity

Free UTC-action:

$F : \mathcal{C} \to \mathrm{Bim}_{\mathrm{fgp}}(A)$ is free iff $\forall c \in \mathcal{C}, \forall \xi \in F(c)$:

$$\inf \left\{ \left\| \sum_1^n a_i^* \rhd \xi \lhd a_i \right\| \;\middle|\; \{a_i\}_1^n \subset A,\ \sum_1^n a_i^* a_i = 1 \right\} = 0.$$

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Theorem ([HPN23])

Let $1 \in A$ be simple, $\mathcal{C} \overset{F}{\curvearrowright} A$ be free and outer, and $\mathbb{B} \in \mathrm{C^*Alg}(\mathcal{C})$. Then $A \rtimes_{F,r} \mathbb{B}$ remains simple.

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Thank you!

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