Equilibrium on Toeplitz extensions of higher dimensional noncommutative tori

Camila F. Sehnem

University of Waterloo

joint work with Z. Afsar, M. Laca and J. Ramagge

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Dimension 2: rotation algebras

- Let $\theta \in \mathbb{R}$. The *rotation algebra* $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries U and V satisfying the relation

$$UV = e^{-2\pi i\theta} VU;$$

the C^*-algebra $\mathcal{A}_\theta$ is also known as *noncommutative torus*.

- $\mathcal{A}_\theta$ can also be described as the crossed product of $\mathrm{C}(\mathbb{T})$ by the automorphism induced by rotation by θ:

$$\varphi(f)(z) := f(e^{-2\pi i\theta} z), \quad f \in \mathrm{C}(\mathbb{T}), z \in \mathbb{T}.$$

- Alternatively, $\mathcal{A}_\theta$ can be viewed as a twisted group C^*-algebra $\mathrm{C}^*(\mathbb{Z}^2, \sigma_\Theta)$, where the 2-cocycle $\sigma_\Theta \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ is given by

$$\sigma_\Theta(x, y) := e^{-\pi i \langle x \,|\, \Theta y \rangle}$$

where Θ is the antisymmetric matrix $(a_{ij})_{i,j}$ such that $a_{12} = \theta$.

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Motivation from noncommutative solenoids

- Solenoids are inverse limits of tori, and so the algebra of continuous functions on a solenoid is a direct limit of $\mathrm{C}(\mathbb{T})$;
- Latremoliere–Packer, '18 defined *noncommutative solenoids*: certain twisted group algebras of abelian discrete groups $\mathbb{Q}_N \times \mathbb{Q}_N$, where $\mathbb{Q}_N$ has a solenoid as its Pontryagin dual.
- A noncommutative solenoid is a direct limit of noncommutative tori $\varinjlim(\mathcal{A}_{\theta_n}, \psi_n)$;
- Toeplitz noncommutative solenoid by Brownlowe–Hawkins–Sims, '17: replaced one unitary generator of $\mathcal{A}_{\theta_n}$ by an isometry and considered the KMS state structure of the resulting direct limit.

Higher-rank noncommutative tori

- The algebra of continuous functions $\mathrm{C}(\mathbb{T}^n)$ is (isomorphic to) the universal C^*-algebra generated by *n commuting* unitaries, where $\mathbb{T}$ is the unit circle; equivalently, $\mathrm{C}(\mathbb{T}^n)$ is the group C^*-algebra $\mathrm{C}^*(\mathbb{Z}^n)$.
- Let $\Theta = (\theta_{i,j})$ be an $n \times n$ antisymmetric matrix with real coefficients. The *n-dimensional noncommutative torus* $\mathcal{A}_\Theta$ is the universal C^*-algebra generated by unitaries $U_1, \ldots, U_n$ satisfying

$$U_j U_k = e^{-2\pi i \theta_{j,k}} U_k U_j \qquad j, k = 1, 2, \ldots, n.$$

- The matrix Θ determines a 2-cocycle $\sigma_\Theta \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ by

$$\sigma_\Theta(x, y) := e^{-\pi i \langle x \,|\, \Theta y \rangle},$$

and $\mathcal{A}_\Theta$ is isomorphic to the twisted group C^*-algebra $\mathrm{C}^*(\mathbb{Z}^n, \sigma_\Theta)$.

A Toeplitz extension of a noncommutative torus

Let $n = k + d$ and $\Lambda \in M_{k,d}(\mathbb{R})$. Afsar–an Huef–Raeburn–Sims, '19 defined the *higher-rank Toeplitz noncommutative torus* B_Λ as the universal C^*-algebra generated by a *Nica-covariant* isometric representation $V\colon \mathbb{N}^k \to B_\Lambda$ and a unitary representation $U\colon \mathbb{Z}^d \to B_\Lambda$ satisfying relations encoded in Λ.

- A vector $r \in (0, \infty)^k$ determines a strongly continuous one-parameter automorphism group $\{\alpha_t^r \mid t \in \mathbb{R}\}$ of B_Λ characterised by

$$\alpha_t^r(V_p) = e^{i\langle p \mid r\rangle t} V_p, \qquad \alpha_t^r(U_x) = U_x \quad (p \in \mathbb{N}^k, x \in \mathbb{Z}^d).$$

- AaHRS studied the KMS state structure for the dynamics α^r, and also considered certain direct limits $\varinjlim(B_{\Lambda_n}, \psi_n)$, called *higher-rank noncommutative solenoids*.

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An approach using twisted semigroup C^*-algebras of $\mathbb{N}^n$

Let $n = k + d$ and let $\Theta = (\theta_{i,j})$ be an $n \times n$ antisymmetric matrix with real coefficients, and let $\sigma_\Theta \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ be given by $\sigma_\Theta(x, y) := e^{-\pi i \langle x \mid \Theta y \rangle}$. Let $\{\delta_q \mid q \in \mathbb{N}^n\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{N}^n)$. Then for each $p \in \mathbb{N}^n$, the map L_p^σ defined by

$$L_p^\sigma \delta_q := \sigma_\Theta(p, q)\delta_{p+q}, \qquad (q \in \mathbb{N}^n)$$

induces an isometric σ_Θ-representation of $\mathbb{N}^n$ such that

$$(L_p^\sigma)^*(L_q^\sigma) = \overline{\sigma(p, (p \vee q) - p)}\sigma(q, (p \vee q) - q) L^\sigma_{-p+(p\vee q)} {L^\sigma_{-q+(p\vee q)}}^*.$$

Definition (Afsar-Laca-Ramagge-S., '21)

The n-dimensional *Toeplitz noncommutative torus* associated to Θ is the twisted semigroup C^*-algebra $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$.

Relationship to AaHRS's Toeplitz noncommutative tori

Proposition (Afsar-Laca-Ramagge-S., 21)

Suppose k and d are nonnegative integers with $n = k + d$. For each rectangular $k \times d$ matrix $\Lambda \in M_{k,d}([0,\infty))$ define $\Theta \in M_{k+d}(\mathbb{R})$ by

$$\Theta := \left[\begin{array}{c|c} 0_{k\times k} & \Lambda \\ \hline -\Lambda^T & 0_{d\times d} \end{array}\right].$$

Then the C^*-algebra B_Λ associated to Λ by AaHRS is canonically isomorphic to the quotient of $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ by the ideal generated by the projections $1 - L_{e_j}L_{e_j}^*$ for $j = k+1, k+2, \ldots, k+d$.

$\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ as an extension of $\mathcal{A}_\Theta$

Proposition

The map that sends an isometry $L_p \in \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ to the unitary $\lambda_p \in \mathrm{C}^*_r(\mathbb{Z}^n, \tilde{\sigma}_\Theta) \cong \mathcal{A}_\Theta$ determines an exact sequence

$$0 \to \mathcal{I} \to \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta) \to \mathcal{A}_\Theta \to 0,$$

where $\mathcal{I}$ is the ideal of $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ generated by the projections

$$\{1 - L_{e_j} L_{e_j}^* \mid j = 1, \ldots, n\}.$$

$\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ can also be described as the universal C^*-algebra generated by isometries $\{w_j \mid j \in \{1, \ldots, n\}\}$ satisfying the relations

$$\begin{cases} w_j w_k = e^{-2\pi i \theta_{j,k}} w_k w_j & j, k = 1, 2, \ldots, n; \\ w_j^* w_k = e^{2\pi i \theta_{j,k}} w_k w_j^* & j \neq k. \end{cases}$$

The dynamics on $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$

Let $r \in \mathbb{R}^n$ and consider the dynamics $\alpha^r : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta))$,

$$\alpha_t^r(L_p) = e^{i\langle p \,|\, r\rangle t} L_p \qquad (p \in \mathbb{N}^n, t \in \mathbb{R}).$$

Let $\beta \in \mathbb{R}$. A state φ of $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ is a KMS_β state for $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta, \alpha^r)$ if it satisfies the KMS_β condition for A α^r-analytic and $B \in \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$

$$\varphi(AB) = \varphi(B\alpha_{i\beta}^r(A))$$

If φ is a KMS_β state for α^r, then $e^{-\langle p \,|\, r\rangle \beta} = \varphi(L_p L_p^*) \leq 1$. So:

- If $\beta > 0$ and $r_j < 0$ for some $j \in \{1, 2, \ldots, n\}$, then there is no KMS_β state; similarly, if $\beta < 0$ and $r_j > 0$ for some $j \in \{1, 2, \ldots, n\}$, then there is no KMS_β state.

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On the vanishing coordinates of r

We work with $\beta > 0$ and so we fix a vector $r \in [0, \infty)^n$, and simply write α instead of α^r.

Lemma

Let φ_β be a KMS_β state of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$. Let $(\mathcal{H}_{\varphi_\beta}, \pi_{\varphi_\beta})$ denote the associated GNS representation. If $p \in \mathbb{N}^n$ and $\langle p \mid r \rangle = 0$, then $\pi_{\varphi_\beta}(L_p)$ is unitary. Moreover, φ_β factors through the quotient of $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$ modulo the ideal generated by the projections $\{1 - L_{e_j} L_{e_j}^* \mid r_j = 0\}$.

- WLOG we assume that all the nonzero coordinates of r appear at the beginning, so that $r = (r_1, r_2, \ldots r_k, 0_d)$ with $r_j > 0$ for $j = 1, \ldots, k$.

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The gauge action on $\mathcal{T}_r(\mathbb{N}^n, \sigma)$

There is a canonical *gauge action* γ of $\mathbb{T}^n$ on $\mathcal{T}_r(\mathbb{N}^n, \sigma)$ given by $\gamma_z(L_p) = z^p L_p$ where $z^p := \prod_{i=1}^n z_i^{p_i}$. This yields a faithful conditional expectation

$$E\colon \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta) \to \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)^\gamma = \overline{\operatorname{span}}\{L_p L_p^* \mid p \in \mathbb{N}^n\}.$$

Proposition

Let $n = k + d$ with $k, d \in \mathbb{N}$ and let $E^{(k)} := E^{\mathbb{T}^k \times \{1_d\}}$ denote the conditional expectation associated to the restriction of the gauge action of $\mathbb{T}^n$ to the closed subgroup $\mathbb{T}^k \times \{1_d\}$. Then

(1) $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta) = \overline{\operatorname{span}}\{L_p L_x L_y^* L_q^* \mid p, q \in \mathbb{N}^k \times 0_d,\ x, y \in 0_k \times \mathbb{N}^d\}$;

(2) $E^{(k)}(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)) = \overline{\operatorname{span}}\{L_p L_x L_y^* L_p^* \mid p \in \mathbb{N}^k \times 0_d,\ x, y \in 0_k \times \mathbb{N}^d\}$.

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A characterisation of KMS_β states of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$

Proposition

Let $n = k + d$ with $k, d \in \mathbb{N}$ and let α be the dynamics determined by $r = (r_1, \ldots, r_k, 0_d)$. Let $0 < \beta < \infty$ and suppose that φ is a KMS_β state of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$. Then φ restricts to a trace on the C^*-subalgebra $\mathrm{C}^*(L_x : x \in 0_k \times \mathbb{N}^d)$ and satisfies

$$\varphi(L_p L_x L_y^* L_q^*) = \delta_{p,q} e^{-\beta\langle p \mid r\rangle} \varphi(L_x L_y^*)$$

for all $p, q \in \mathbb{N}^k \times 0_d$ and $x, y \in 0_k \times \mathbb{N}^d$, where $\delta_{p,q}$ is the Kronecker delta.

In particular this implies that KMS_β states factor through the conditional expectation $E^{(k)}$.

The restriction of σ_Θ to $0_k \times \mathbb{N}^d$

Consider the projection $Q := \prod_{j=1}^{k}(1 - L_{e_i} L_{e_i}^*)$ in $\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)$. Then:

(1) $QL_p = 0 = L_p^* Q$ for every $p \in \mathbb{N}^k \times 0_d \setminus \{0\}$;

(2) $QL_p^* L_x L_y^* L_p = L_p^* L_x L_y^* L_p Q$ for every $x, y \in 0_k \times \mathbb{N}^d$ and $p \in \mathbb{N}^k \times 0_d$;

(3) $Q\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)Q = \overline{\mathrm{span}}\{QL_x L_y^* Q \mid x, y \in 0_k \times \mathbb{N}^d\}$.

Lemma

Let Θ_d denote the lower right $d \times d$ corner of Θ. Then $\mathrm{C}^*(L_x : x \in 0_k \times \mathbb{N}^d)$ is canonically isomorphic to the Toeplitz noncommutative torus $\mathcal{T}_r(\mathbb{N}^d, \sigma_{\Theta_d})$, and the map $\rho_Q \colon \mathrm{C}^*(L_x : x \in 0_k \times \mathbb{N}^d) \to Q\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)Q$ given by the compression $X \mapsto QXQ$ is an isomorphism.

KMS_β states from traces on the corner

Suppose that φ is a KMS_β state for $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$. Let $Q = \prod_{j=1}^{k}(1 - L_{e_i} L_{e_i}^*)$. Then $\varphi(Q) > 0$ and $\varphi(Q)^{-1} = Z(\beta)$, where

$$Z(\beta) := \sum_{p \in \mathbb{N}^k \times 0_d} e^{-\beta \langle p \,|\, r \rangle}.$$

Proposition

Let $\beta > 0$. For each tracial state ω of the corner $Q\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)Q$, define

$$T_\beta(\omega)(X) := \frac{1}{Z(\beta)} \sum_{l \in \mathbb{N}^k \times 0_d} e^{-\beta \langle l \,|\, r \rangle} \omega(Q L_l^* X L_l Q), \qquad X \in \mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta).$$

Then T_β is an affine weak homeomorphism of the tracial state space of the corner onto the KMS_β state space of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$.*

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KMS_β states from traces on a noncommutative torus

Proposition

Let $\rho_Q\colon \mathrm{C}^*(L_x \mid x \in 0_k \times \mathbb{N}^d) \to Q\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta)Q$ be the compression by the projection Q. For each tracial state τ of $\mathcal{T}_r(\mathbb{N}^d, \sigma_{\Theta_d})$ there is a KMS_β state of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$ determined by

$$T_\beta(\tau \circ \rho_Q^{-1})(L_p L_x L_y^* L_q^*) = \delta_{p,q}\, \tau(L_x L_y^*) \prod_{j=1}^{k} \frac{e^{-\beta r_j p_j}(1 - e^{-\beta r_j})}{1 - e^{-\beta r_j + 2\pi i \langle \Theta(x-y) \mid e_j \rangle}},$$

where $x, y \in 0_k \times \mathbb{N}^d \cong \mathbb{N}^d$. The map $\tau \mapsto T_\beta(\tau \circ \rho_Q^{-1})$ is an affine weak homeomorphism of the tracial state space of $\mathcal{T}_r(\mathbb{N}^d, \sigma_{\Theta_d})$ onto the simplex of KMS_β states of $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha)$.*

Traces on higher-rank noncommutative tori

Let D be an antisymmetric, real, $d \times d$ matrix.

- The *degeneracy index* of D is $m := \operatorname{rank} H$ where H is the subgroup

$$H := \{x \in \mathbb{Z}^d \mid \langle x \mid Dy \rangle \in \mathbb{Z} \text{ for all } y \in \mathbb{Z}^d\}.$$

- $\mathcal{A}_D$ is simple iff D is nondegenerate (i.e. $m = 0$) (Slawny; Phillips).
- There is an action of a compact group

$$\Lambda = \mathbb{Z}_{a_1} \times \cdots \mathbb{Z}_{a_m} \times \mathbb{T}^{d-m} \subset \mathbb{T}^d$$

  whose fixed-point algebra is the center $\mathrm{Z}(\mathcal{A}_D)$ of $\mathcal{A}_D$.
- $\mathrm{Z}(\mathcal{A}_D)$ is isomorphic to $\mathrm{C}(\mathbb{T}^m)$.

Proposition

Let $E^\Lambda \colon \mathcal{A}_D \to \mathrm{Z}(\mathcal{A}_D)$ be the canonical conditional expectation associated to the action of Λ. Then the map $\omega \mapsto \omega \circ E^\Lambda$ is an affine homeomorphism of the state space of $\mathrm{Z}(\mathcal{A}_D)$ onto the space of tracial states of $\mathcal{A}_D$.

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- $\mathcal{A}_D$ is simple iff D is nondegenerate (i.e. $m = 0$) (Slawny; Phillips).
- There is an action of a compact group

$$\Lambda = \mathbb{Z}_{a_1} \times \cdots \mathbb{Z}_{a_m} \times \mathbb{T}^{d-m} \subset \mathbb{T}^d$$

whose fixed-point algebra is the center $\mathrm{Z}(\mathcal{A}_D)$ of $\mathcal{A}_D$.

- $\mathrm{Z}(\mathcal{A}_D)$ is isomorphic to $\mathrm{C}(\mathbb{T}^m)$.

Proposition

Let $E^\Lambda \colon \mathcal{A}_D \to \mathrm{Z}(\mathcal{A}_D)$ be the canonical conditional expectation associated to the action of Λ. Then the map $\omega \mapsto \omega \circ E^\Lambda$ is an affine homeomorphism of the state space of $\mathrm{Z}(\mathcal{A}_D)$ onto the space of tracial states of $\mathcal{A}_D$.

The main theorem: a parametrisation of KMS_β states

Theorem (Afsar-Laca-Ramagge-S.,21)

Let $\Theta = \left[\begin{array}{c|c} \Theta_k & \Lambda \\ \hline -\Lambda^T & \Theta_d \end{array}\right]$ *and let* m *be the degeneracy index of* Θ_d*. Then there is an affine weak* homeomorphism of the space* $M_1(\mathbb{T}^m)$ *of probability measures on* $\mathbb{T}^m$ *onto the space of* KMS_β *states of* $(\mathcal{T}_r(\mathbb{N}^n, \sigma_\Theta), \alpha^r)$*. If* $\{p_1, \dots, p_d\}$ *is a basis for* $\mathbb{Z}^d$ *such that* $\{a_1 p_1, \dots, a_m p_m\}$ *is a basis for* H *then the homeomorphism can be chosen so that the extremal* KMS_β *state* $\varphi_{\beta,z}$ *associated to the unit point mass at* $z \in \mathbb{T}^m$ *is given by*

$$\varphi_{\beta,z}(L_p L_x L_y^* L_q^*) = \delta_{p,q}\,[x - y \in H]\,\lambda_{x-y}\, z^c \prod_{j=1}^{k} \frac{e^{-\beta r_j p_j}(1 - e^{-\beta r_j})}{1 - e^{-\beta r_j + 2\pi i \langle \Theta(x-y) \,|\, e_j \rangle}}$$

where $c = (c_1, \dots, c_m)$ *is the vector of coefficients of* $x - y$ *with respect to the basis* $\{a_1 p_1, \dots, a_m p_m\}$ *of* H *and* $\lambda_{x-y} \in \{-1, 1\}$*.*

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Thanks!