# Semiclassical Analysis and Noncommutative Geometry

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# Spectral Triples

**Definition (Baaj-Julg, Connes)**

A $p$-summable spectral triple $(\mathscr{A}, \mathscr{H}, D)$ consists of

- A Hilbert space $\mathscr{H}$.
- A (unital) $*$-subalgebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$.
- A selfadjoint (unbounded) operator $D$ on $\mathscr{H}$ s.t.

$$[D, a] \in \mathscr{L}(\mathscr{H}) \quad \forall a \in \mathscr{A},$$

$$\lambda_j(|D|^{-1}) = \mathrm{O}\left(j^{-\frac{1}{p}}\right).$$

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### Remarks

1. $\mathrm{Tr}[|D|^{-q}] < \infty$ for all $q > p$.
2. Degree of summability $\simeq$ dimension of $(\mathscr{A}, \mathscr{H}, D)$.

# Spectral Triples

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$(M^n, g)$ closed Riemannian manifold.

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**Example (Dirac Spectral Triple)**

$$\left(C^\infty(M), L^2(M, \displaystyle{\not}S), \displaystyle{\not}D\right),$$

where $\displaystyle{\not}D$ is the Dirac operator acting on the spinor bundle $\displaystyle{\not}S$.

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**Remark**

These spectral triples are $n$-summable.

# Quantized Calculus (Connes)

| **Classical** | **Quantum** (Connes) |
|---|---|
| Complex variable | Operator on Hilbert space $\mathcal{H}$ |
| Real variable | Selfadjoint operator on $\mathcal{H}$ |
| Infinitesimal variable | Compact operator on $\mathcal{H}$ |
| Infinitesimal of order $\alpha$ | Compact operator s.t. $\lambda_j(\|T\|) = O(j^{-\alpha})$ |
| Integral $\int f(x)dx$ | NC integral $⨍ T$ |

Here $\lambda_0(|T|) \geq \lambda_1(|T|) \geq \cdots$ are the eigenvalues of $|T|$.

# NC Integral

## Setup

- $T$ is an infinitesimal of order $1$, i.e., $\lambda_j(|T|) = O(j^{-1})$.
- $\{\lambda_j(T)\}$ eigenvalue sequence s.t. $|\lambda_0(T)| \geq |\lambda_1(T)| \geq \cdots$.

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## Definition (Connes)

1. We say that $T$ is measurable if
$$⨍ T := \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \text{ exists.}$$
2. $⨍ T$ is called the NC integral of $T$.

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## Proposition (Dixmier, Connes, Lord-Sukochev-Zanin, RP)

1. *The measurable operators form a vector space which is conjugation-invariant.*
2. $T \to ⨍ T$ *is a linear trace that vanishes on infinitesimal operators of order* $> 1$.

# Connes' Integration Formula

**Setup**

- $(M^n, g)$ closed Riemannian mfld..
- $\Delta_g : C^\infty(M) \to C^\infty(M)$ Laplacian.
- $\nu_g(x)$ Riemannian measure (i.e., $\nu_g(x) = \sqrt{g(x)}dx$).

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## Fact

$(C^\infty(M), L^2(M), \sqrt{\Delta_g})$ *is a spectral triple.*

## Theorem (Connes' Integration Formula '88)

*For every* $f \in C^\infty(M)$,

$$\begin{aligned} ⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} &= \operatorname{Res}_{z=n} \operatorname{Tr}\left[f\Delta_g^{-\frac{z}{2}}\right] \\ &= \lim_{t\to 0^+} t^{\frac{n}{2}} \operatorname{Tr}\left[fe^{-t\Delta_g}\right] \\ &= \int_M f(x) d\nu_g(x). \end{aligned}$$

# Relationship with Weyl's Laws

## Setup

- $A^* = A$ with $\lambda_j(|A|) = \mathrm{O}(j^{-1})$.
- $\lambda_j^{\pm}(A)$ are the positive/negative eigenvalues of $A$, i.e.,

$$\lambda_j^{\pm}(A) = \lambda_j(A_{\pm}), \quad A_{\pm} = \frac{1}{2}(|A| \pm A).$$

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## Proposition

*Assume that*

$$\lim_{j\to\infty} j\lambda_j^{\pm}(A) \text{ both exist.}$$

*Then $A$ is measurable, and*

$$⨍ A = \lim_{j\to\infty} j\lambda_j^{+}(A) - \lim_{j\to\infty} j\lambda_j^{-}(A).$$

# Connes' Integration Formula. Stronger Form

**Theorem (Birman-Solomyak '70s)**

*If* $f \in C^\infty(M, \mathbb{R})$, *then*

$$\lim_{j \to \infty} j^{\frac{q}{n}} \lambda_j^{\pm}\left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}}\right) = \left(\int_M f_{\pm}(x)^{\frac{n}{q}} d\nu_g(x)\right)^{\frac{q}{n}}.$$

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For $q = n$ we obtain:

**Corollary (Connes' Integration Formula)**

1. *If* $f \in C^\infty(M, \mathbb{R})$, $f \geq 0$, *then*

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = \lim_{j\to\infty} j\lambda_j(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}) = \int_M f(x) d\nu_g(x).$$

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2. *By linearity, for all* $f \in C^\infty(M, \mathbb{R})$, *we get*
$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = \int_M f(x) d\nu_g(x).$$

**Notation**

If $H$ is bounded from below withs discrete negative spectrum, then

$$N^-(H) := \#\{\text{negative eigenvalues w/ multiplicity}\}.$$

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## Remark

For Schrödinger operators $H = \Delta + V$ on $\mathbb{R}^n$ this corresponds to the number of "bound states".

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**Theorem (Semiclassical Weyl's law)**

*For any $V \in C(M, \mathbb{R})$, we have*

$$\lim_{h \to 0^+} h^n N^{-}(h^2 \Delta_g + V) = \int_M V_{-}(x)^{\frac{n}{2}} d\nu_g(x).$$

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*More generally, given any $q > 0$,*

$$\lim_{h \to 0^+} h^n N^{-}\left(h^{2q} \Delta_g^q + V\right) = \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x).$$

# Birman-Schwinger Principle

## Setup

- Unbounded operator $H_0 \geq 0$ on $\mathscr{H}$.
- Operator $V^* = V$ s.t. $\lambda_j(|H_0^{-1/2} V H_0^{-1/2}|) = O(j^{-1/p})$.

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## Theorem (Birman-Schwinger Principle)

*We have*

$$\lim_{h \to 0^+} h^{2p} N^-\left(h^2 H_0 + V\right) = \lim_{j \to \infty} j \lambda_j^-\left(H_0^{-1/2} V H_0^{-1/2}\right)^p$$
$$= \int\!\!\!\!\!\!- \left(H_0^{-1/2} V H_0^{-1/2}\right)_-^p.$$

## Summary

Eigenvalue asymptotics for compact operators yield:

- Semiclassical Weyl's laws (Birman-Schwinger principle).
- Stronger form of Connes' integration formula.

**Setup**

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## Theorem (McDonald-Sukochev-Zanin '22)

*Assume the following:*

- $p > 2$.
- Lipschitz regularity: $[|D|, a] \in \mathscr{L}(\mathscr{H})$ *for all* $a \in \mathscr{A}$.
- Tauberian condition: *For all* $a \in \overline{\mathscr{A}}$, $a \geq 0$,

$$\mathrm{Res}_{z=p} \,\mathrm{Tr}\left[a^z |D|^{-z}\right] \text{ exists.}$$

# SC Analysis on Spectral Triples

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*Then, for every* $V^* = V \in \overline{\mathscr{A}}$,

$$\lim_{h \to 0^+} h^p N^-\left(h^2 D^2 + V\right) = ⨍ |D|^{-\frac{p}{2}} V_-^{\frac{p}{2}} |D|^{-\frac{p}{2}}$$

$$= \mathrm{Res}_{z=p} \operatorname{Tr}\left[V_-^{z/2} |D|^{-z}\right].$$

**Question**

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**Question (Connes)**

Can we relate the MSZ Tauberian condition to more standard Tauberian conditions?

# Main Results

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## Remark

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## Notation

If $a \in \overline{\mathscr{A}}$, $a > 0$ (positive + invertible), then

$$\lambda_0(aD^2a) \leq \lambda_1(aD^2a) \leq \cdots$$

are the (positive) eigenvalues of $aD^2a$.

## Main Results – Spectral Asymptotics

**Theorem (RP '23)**

*Assume that, for all* $a \in \mathscr{A}$, $a > 0$,

$$\lim_{j\to\infty} j^{-\frac{1}{p}} \lambda_j\big(aD^2a\big) = \tau\big[a^{-p}\big]^{-\frac{2}{p}},$$

*where* $\tau : \overline{\mathscr{A}} \to \mathbb{C}$ *is a given positive linear map. Then:*

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2. Semiclassical Weyl's law. *Given any* $q > 0$, *for all* $V = V^* \in \overline{\mathscr{A}}$,

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3. Integration formula. *For all* $a \in \overline{\mathscr{A}}$,

$$⨍ |D|^{-\frac{p}{2}} a |D|^{-\frac{p}{2}} = \tau[a].$$

# Idea of Proof

**Lemma (Birman-Solomyak '70s)**

*If* $a_\ell \to a$ *in* $\overline{\mathscr{A}}$ *with* $a_\ell^* = a_\ell$, *then*

$$\lim_{j\to\infty} j^{\frac{q}{p}} \lambda_j^{\pm}\left(|D|^{-\frac{q}{2}} a |D|^{-\frac{q}{2}}\right) = \lim_{\ell\to\infty}\lim_{j\to\infty} j^{\frac{q}{p}} \lambda_j^{\pm}\left(|D|^{-\frac{q}{2}} a_\ell |D|^{-\frac{q}{2}}\right).$$

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**Lemma (Sukochev-Zanin, RP)**

*Given any* $q > 0$ *and* $a = a^* \in \overline{\mathscr{A}}$,

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**Lemma (RP)**

*Given any $q > 0$ and $a \in \mathscr{A}$, $a > 0$,*

$$\lim_{j\to\infty} j^{\frac{q}{p}} \lambda_j\left(|D|^{-\frac{q}{2}} a |D|^{-\frac{q}{2}}\right) = \left[\lim_{j\to\infty} j^{-\frac{2}{p}} \lambda_j\left(a^{-\frac{1}{q}} D^2 a^{-\frac{1}{q}}\right)\right]^{-\frac{q}{2}},$$

# Main Results – Tauberian Theorem

**Theorem (Tauberian Theorem; RP '23)**

*Given any* $a \in \mathscr{A}$, $a > 0$, *we have*

$$\begin{aligned}\lim_{j\to\infty} j\lambda_j\left(aD^2a\right)^{-\frac{p}{2}} &= \operatorname{Res}_{z=p} \operatorname{Tr}\left[a^{-z}|D|^{-z}\right] \\ &= \operatorname{Res}_{z=p} \operatorname{Tr}\left[a^{-p}|D|^{-z}\right] \\ &= \lim_{t\to 0^+} t^{\frac{p}{2}} \operatorname{Tr}\left[a^{-p}e^{-tD^2}\right],\end{aligned}$$

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*provided any of the functionals on the r.h.s. exist.*

**Remark**

- The last two Tauberian conditions are very natural conditions for spectral triples (e.g., Connes-Moscovici's local index formula, spectral action).
- They are satisfied in numerous examples.

# Main Results

**Corollary (RP)**

*Assume that, for all* $a \in \mathscr{A}$,

$$\operatorname{Res}_{z=p} \operatorname{Tr}\left[a|D|^{-z}\right] = \tau[a], \quad \text{or} \quad \lim_{t\to 0^+} t^{\frac{p}{2}} \operatorname{Tr}\left[ae^{-tD^2}\right] = \tau[a].$$

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# Riemannian Case Revisited

**Reminder**

If $(M^n, g)$ is a closed Riemannian manifold, then

$$\left(C^\infty(M), L^2(M), \sqrt{\Delta_g}\right)$$

is an $n$-summable spectral triple.

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$$\left(C^\infty(M), L^2(M), \sqrt{\Delta_g}\right)$$

is an $n$-summable spectral triple.

**Theorem (Minakshisundaram-Pleijel CJM '49)**

1. *Given any $f \in C^\infty(M)$, we have*

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2. *For $f = 1$ this gives the classical Weyl's law,*

$$\lambda_j(\Delta_g) \sim \left(\frac{j}{\mathrm{Vol}_g(M)}\right)^{\frac{2}{n}} \quad \text{as } j \to \infty.$$

# Riemannian Case Revisited

## Corollary

1. Birman-Solomyak's asymptotics. *Given any* $q > 0$, *for all* $f \in C(M, \mathbb{R})$,

$$\lim_{j \to \infty} j^{\frac{q}{n}} \lambda_j^{\pm}\left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}}\right) = \left(\int_M f_{\pm}(x)^{\frac{n}{q}} d\nu_g(x)\right)^{\frac{q}{n}}.$$

2. Semiclassical Weyl's law. *Given any* $q > 0$, *for all* $V \in C(M, \mathbb{R})$,

$$\lim_{h \to 0^+} h^n N^-\left(h^{2q} \Delta_g^q + V\right) = \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x).$$

3. Connes' integration formula. *For all* $f \in C(M)$,

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = \int_M f(x) d\nu_g(x).$$

# Noncommutative Torus

## Setup

- $\theta = (\theta_{jk})$ real anti-symmetric $n \times n$-matrix.

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The noncommutative torus $\mathbb{T}^n_\theta$ is the NC space whose $C^*$-algebra $C(\mathbb{T}^n_\theta)$ is generated by unitaries $U_1, \ldots, U_n$ such that

$$U_k U_j = e^{2i\pi\theta_{jk}} U_j U_k.$$

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## Remarks

1. For $\theta = 0$ we get the $C^*$-algebra $C(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the ordinary torus.
2. A dense basis of $C(\mathbb{T}^n_\theta)$ is given by the monomials,

$$U^m := U_1^{m_1} \cdots U_n^{m_n}, \qquad m = (m_1, \ldots, m_n) \in \mathbb{Z}^n.$$

# $L^2$-Space

**Definition**

$\tau : C(\mathbb{T}^n_\theta) \to \mathbb{C}$ is the faithful positive trace defined by

$$\tau_0(1) = 1, \qquad \tau_0\left(U^m\right) = 0, \quad m \neq 0.$$

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$L^2(\mathbb{T}^n_\theta)$ is the Hilbert space completion with respect to the pre-inner product $\langle u|v\rangle := \tau_0(v^* u)$, $u, v \in C(\mathbb{T}^n_\theta)$.

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**Proposition (GNS Representation)**

*The action of $C(\mathbb{T}^n_\theta)$ on itself by left-multiplication extends to a $*$-representation in $L^2(\mathbb{T}^n_\theta)$.*

**Definition**

1. The canonical derivations $\partial_1, \ldots, \partial_n$ are given by

$$\partial_j(U_j) = iU_j, \qquad \partial_j(U_k) = 0, \quad k \neq j.$$

# Smooth Structure of $\mathbb{T}^n_\theta$

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$$C^\infty(\mathbb{T}^n_\theta) := \Big\{ u = \sum_{m \in \mathbb{Z}^n} u_m U^m, \ (u_m) \in \mathcal{S}(\mathbb{Z}^n) \Big\}.$$

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## Proposition

*$C^\infty(\mathbb{T}^n_\theta)$ is a Fréchet $*$-algebra and is closed under holomorphic functional calculus.*

**Definition**

The Laplacian $\Delta : C^\infty(\mathbb{T}_\theta^n) \to C^\infty(\mathbb{T}_\theta^n)$ is given by

$$\Delta := \partial_1^2 + \cdots + \partial_n^2.$$

**Proposition**

1. *$\Delta$ is an essentially selfadjoint operator such that*

$$\Delta\left(U^m\right) = |m|^2 U^m \qquad \forall m \in \mathbb{Z}^n.$$

2. *In particular, $\Delta$ is isospectral to the Laplacian on the ordinary torus $\mathbb{T}^n$.*

# Spectral Triple – Integration Formula

**Proposition**

*The triple*

$$\left(C^\infty(\mathbb{T}^n_\theta), L^2(\mathbb{T}^n_\theta), \sqrt{\Delta}\right)$$

*is an $n$-summable spectral triple.*

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**Theorem (Integration Formula; McDonald-Sukochev-Zanin, RP)**

*For every $a \in C(\mathbb{T}^n_\theta)$, we have*

$$⨍ \Delta^{-\frac{n}{4}} a \Delta^{-\frac{n}{4}} = \tau_0[a].$$

**Conjecture (McDonald+RP '21)**

*Given any* $q > 0$, *for all* $V = V^* \in C^\infty(\mathbb{T}^n_\theta)$,

$$\lim_{h\to 0^+} N^-\big(h^{2q}\Delta^q + V\big) = \tau_0\Big[(V_-)^{\frac{n}{2q}}\Big].$$

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**Remark**

- The conjecture is proved for $q = 1$ and $n \geq 3$ by McDonald-Suckochev-Zanin as a consequence of their semiclassical Weyl's laws for spectral triples.

# Semiclassical Weyl's Law on $\mathbb{T}^n_\theta$

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## Remark

- The conjecture is proved for $q = 1$ and $n \geq 3$ by McDonald-Suckochev-Zanin as a consequence of their semiclassical Weyl's laws for spectral triples.
- Their approach does not to allow us to get a semiclassical Weyl's law for NC 2-tori.

# Heat-Trace Asymptotics

**Lemma**

*Let* $a \in C^\infty(\mathbb{T}^n_\theta)$. *As* $t \to 0^+$, *we have*

$$\mathrm{Tr}\left[ae^{-t\Delta}\right] = \pi^{\frac{n}{2}}\tau_0[a]t^{-\frac{n}{2}} + \mathrm{O}\left(t^{\frac{-(n-1)}{2}}e^{-\frac{\pi^2}{t}}\right).$$

# Heat-Trace Asymptotics – Proof

Proof.

- We have

$$\operatorname{Tr}\left[ae^{-t\Delta}\right] = \sum_{m\in\mathbb{Z}^n} \langle ae^{-t\Delta}U^m|U^m\rangle = \sum_{m\in\mathbb{Z}^n} e^{-t|m|^2}\langle aU^m|U^m\rangle.$$

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- Note that

$$\langle aU^m|U^m\rangle = \tau_0[(U^m)^* aU^m] = \tau_0[aU^m(U^m)^*] = \tau_0[a]$$

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- By Poisson's summation formula,

$$\sum_{k\in\mathbb{Z}} e^{-tk^2} = \sqrt{\frac{\pi}{t}} + \sum_{|k|\geq 1} e^{-\frac{\pi^2k^2}{t}} = \sqrt{\frac{\pi}{t}} + \mathrm{O}\left(e^{-\frac{\pi^2}{t}}\right).$$

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- This gives the result.

□

As a consequence the conjecture with Ed McDonald is true:

**Corollary (RP '23)**

*Given any* $q > 0$, *for all* $V = V^* \in C(\mathbb{T}_\theta^n)$,

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