Notes on regular ideals of C*-algebras

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The gauge action of $\mathbb{T}$ on $C^*(E)$ is given by $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$.

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Theorem (Bates-Pask-Raeburn-Szymański, 2000)
Let E be a directed graph. If J is a gauge-invariant ideal of $C^*(E)$, then $J = I(H(J))$ and $C^*(E)/J \cong C^*(E/J)$. Moreover, if E satisfies Condition (K) then so does E/J.

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Theorem (Brown-Fuller-Pitts-R, 2022)

Let E be a row-finite graph satisfying Condition (L). Let J be a regular, gauge-invariant ideal in $C^*(E)$. Then E/J satisfies Condition (L).

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Proof: The one-to-one correspondence between the ideals of A and open sets in $\mathrm{Prim}(A)$ given by

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Example: If X is compact and Hausdorff then $J \subseteq C(X)$ is regular iff $\{x \in X \,|\, \forall f \in J\; f(x) = 0\}$ is the closure of its interior.

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Lemma: For a vertex w, let $T(w) = \{s(\alpha) \mid \alpha \in E^*, r(\alpha) = w\}$, the sources of paths in E ending at w. Then a gauge-invariant ideal J is regular iff

$$p_w \in J \;\Leftrightarrow\; \forall v \in T(w)\; \exists u \in T(v) \text{ s.t. } p_u \in H(J).$$

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Proof of Theorem

Let $\lambda = e_1 e_2 \cdots e_n$ be a cycle in F. Since E has Condition (L), $\mathcal{E} := \{e \in E^1 : e \text{ is an entrance for } \lambda\} \neq \emptyset$. If λ has no entry in E/J then $\forall v \in s(\mathcal{E}), p_v \in H(J)$. Since J is regular, it follows from the lemma that for any vertex w of λ, $p_w \in J$, contradicting that λ is a cycle in E/J. Hence E/J has Condition (L).

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Barlak-Li (2017, 2019): Any separable and nuclear C^*-algebra containing a Cartan subalgebra satisfies the UCT. K-theory classifies C^*-algebras with finite nuclear dimension that satisfy the UCT.

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C^*-dynamical systems: The IIP used to characterize when the action of a discrete abelian group on a C^*-algebra is topologically free. (Archbold-Spielberg; extended by Sierakowski and Kennedy-Schafhauser)

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Corollary The same result for Cartan pairs.

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Key Lemma: If J and K are ideals with $J^\perp \subseteq K$ then $J^\perp = K$ or $K \cap J \neq \{0\}$.

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(ii) find a non-regular ideal J such that E/J also satisfies (L), where E/J is defined so that $C^*(E/J) = C^*(E)/J$, and
(iii) recall that (L) guarantees that the diagonal subalgebra generated by the path projections $D := C^*(\{s_\alpha s_\alpha^* \mid \alpha \text{ a path in } E\})$ is Cartan.

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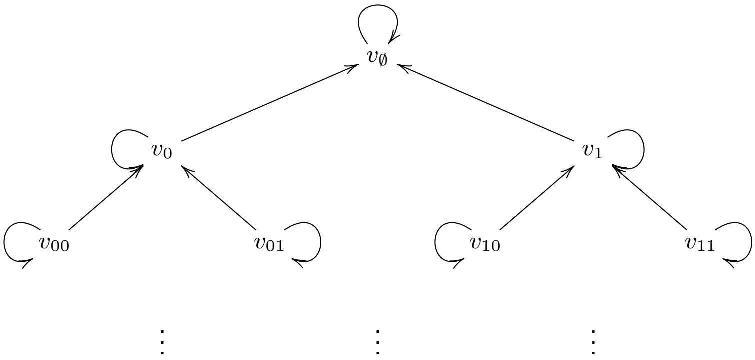

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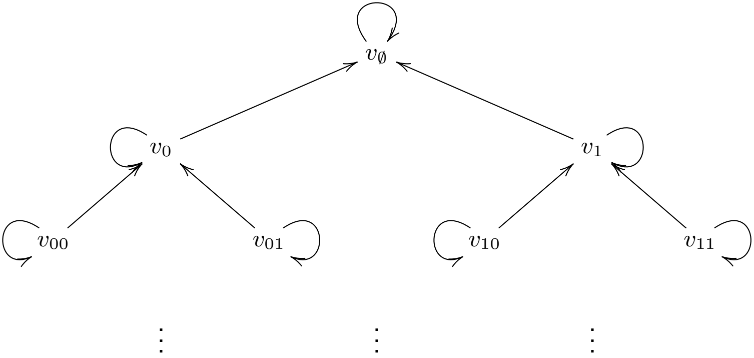
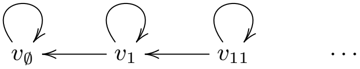


Let $H = T(v_0) \sqcup \left(\bigsqcup_{i \in \mathbb{N}+} T(v_{\underbrace{11\ldots1}_{i}0}) \right)$, where $T(v) := s(r^{-1}(v))$. H is a saturated hereditary set.

In our example above, $E/I(H) = (E^0 \backslash H, s^{-1}(E^0 \backslash H), r, s)$ is the graph



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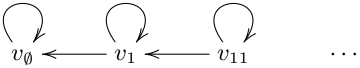

which has Condition (L).

However, $I(H)$ is not regular because for example $w = v_0 \notin H = H(I(H))$ though $\forall v \in T(w)\ \exists u \in T(v) \cap H$ violates the technical lemma.

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Let (A, D) be a regular inclusion with D abelian and assume that D^c, the relative commutant of D in A, is abelian. If (A, D^c) is a Cartan inclusion, then (A, D) has the ideal intersection property if and only if (A, D) has the regular ideal intersection property.

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Let $\Sigma \to G$ be a groupoid twist and $U \subseteq G^{(0)}$ a regular invariant open set. Then $C_r^*(\Sigma_U, G_U)^{\perp\perp}$ is a regular ideal in $C_r^*(\Sigma, G)$. (If G is exact, this is $C_r^*(\Sigma_U, G_U)$.) Moreover, if $C_0(G^{(0)}) \subseteq C_r^*(\Sigma, G)$ has the ideal intersection property then every regular ideal is of this form..

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Corollary If a graph E satisfies (L) then any regular ideal is gauge invariant. (Use the path groupoid with trivial twist.)

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- Thank you for attending today!

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