A central limit theorem for star-generators of the infinite symmetric group, which relates to traceless CCR-GUE matrices

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The talk presents joint work by J. Campbell, C. Köstler and A. Nica
(International J. Math 2022, also available as arXiv:2203.01763).

Notation.

- $S_\infty :=$ group of all finite permutations σ of $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ (thus $\sigma : \mathbb{N} \to \mathbb{N}$ is bijective, and there exists $n_o \in \mathbb{N}$ such that $\sigma(n) = n$ for $n > n_o$).

- Write permutations in cycle notation, e.g. $\sigma = (1, 3, 2)(5, 6)$ takes $1 \to 3 \to 2 \to 1$, $5 \leftrightarrow 6$ and fixes 4 and all $n \geq 7$.
  [Notation "(1)" will refer to the identity permutation.]

- Star-transpositions:
  $\gamma_1 = (1, 2)$, $\gamma_2 = (1, 3), \ldots, \gamma_n = (1, n+1), \ldots$

It is immediate that the γ_n's generate S_∞, which is why they are known as *the star-generators* of S_∞.

A limit theorem proved by Biane.

Theorem (Biane, 1995). *Consider the* $*$*-probability space* $(\mathbb{C}[S_\infty], \varphi)$, *where* φ *is the canonical trace, i.e.* $\varphi : \mathbb{C}[S_\infty] \to \mathbb{C}$ *is linear and has*

$$\varphi(\sigma) = \left\{ \begin{array}{ll} 1, & \text{if } \sigma = (1), \\ 0, & \text{otherwise.} \end{array} \right\}, \quad \sigma \in S_\infty.$$

View the γ_n*'s as centred selfadjoint elements in* $\mathbb{C}[S_\infty]$. *Put*

$$s_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \gamma_i, \quad n \in \mathbb{N}.$$

Then the s_n*'s converge in moments to the semicircle law of Wigner.*

Remark. It is relevant to use the star-transpositions γ_n. If instead of γ_n's we used Coxeter generators $(1,2),(2,3),\ldots,(n,n+1),\ldots$ then the limit law would be Gaussian.

...limit theorem proved by Biane. Recap: if we put

$$s_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \gamma_i, \quad n \in \mathbb{N},$$

then the s_n's converge in moments in $(\mathbb{C}[S_\infty], \varphi)$, and limit law is the semicircle law of Wigner. The latter is best known as $d \to \infty$ limit in results about $d \times d$ random Hermitian matrices (e.g. in the GUE model – to be reviewed below). Its occurrence in Biane's theorem suggests some random matrices should be in the picture!

In this talk: I will present a limit theorem that involves a GUE matrix of size $d \times d$, for fixed $d \in \mathbb{N}$. Then $d \to \infty$ will retrieve the theorem of Biane.

Besides a dimension $d \geq 2$, interesting to also fix some weights $w_1 \geq w_2 \geq \cdots \geq w_d > 0$ with $\sum_{i=1}^{d} w_i = 1$. Then we run into a version of $d \times d$ GUE matrix with entries from the CCR algebra.

OUTLINE OF HOW THE TALK WILL GO:

I. Framework: the character χ, the W^*-prob space $(\mathcal{M}, \mathrm{tr})$, and the law of large numbers.

II. The exchangeable CLT and how it applies to the operators $U(\gamma_n) \in \mathcal{M}$.

III. A digression: CCR analogue for a traceless GUE matrix.

IV. Identification of the limit law in the CLT from Part II.

Notation *(the character χ of S_∞).*
Fix an integer $d \geq 2$ and some weights $w_1 \geq w_2 \geq \cdots \geq w_d > 0$ with $w_1 + \cdots + w_d = 1$.
Classification of Thoma (1964) has an *extremal character* $\chi : S_\infty \to \mathbb{R}$ associated to these weights. We define it like this:

• Denote $\mathrm{p}_n := w_1^n + w_2^n + \cdots + w_d^n, \quad n \in \mathbb{N}$. Get a sequence of numbers $1 = \mathrm{p}_1 > \mathrm{p}_2 > \cdots > \mathrm{p}_n > \cdots > 0$.

• For $\sigma \in S_\infty$ put $\chi(\sigma) := \prod_{\substack{V \text{ orbit of } \sigma, \\ |V| \geq 2}} \mathrm{p}_{|V|}$.

(E.g. $\sigma = (1,3,2)(5,6) \in S_\infty$ has $\chi(\sigma) = \mathrm{p}_2 \cdot \mathrm{p}_3$.)

Turns out that χ is indeed a character of S_∞ (positive definite, normalized, constant on conjugacy classes), and it is moreover an extreme point in the space of characters.

The special case of the block character.

Suppose we picked our weights to be $w_1 = \cdots = w_d = 1/d$. Then $\mathsf{p}_n = (1/d)^{n-1}$, $n \in \mathbb{N}$. The formula defining χ comes out as

$$\chi(\sigma) = \prod_{\substack{V \text{ orbit of } \sigma, \\ |V| \geq 2}} (1/d)^{|V|-1}.$$

Easy to check: this amounts to

$$\chi(\sigma) = (1/d)^{||\sigma||}, \quad \sigma \in S_\infty, \text{ where}$$

$$||\sigma|| := \min \left\{ m \;\middle|\; \begin{array}{l} \sigma \text{ can be written as a} \\ \text{product of } m \text{ transpositions} \end{array} \right\}.$$

E.g. $\sigma = (1, 3, 2)(5, 6)$ has $\chi(\sigma) = (1/d)^2 \cdot (1/d)^1 = (1/d)^3$, matching the fact that $||\sigma|| = 3$.

This special case of χ is called *block character* of S_∞.

The W^*-probability space $(\mathcal{M}, \mathrm{tr})$.

$\to$ Let $U : S_\infty \to B(\mathcal{H})$ be the GNS representation of χ. That is:

- Have map $S_\infty \ni \sigma \mapsto \widehat{\sigma} \in \mathcal{H}$, such that $\mathrm{span}\{\widehat{\sigma} \mid \sigma \in S_\infty\}$ is dense in $\mathcal{H}$ and such that $\langle \widehat{\sigma}, \widehat{\tau} \rangle = \chi(\sigma\tau^{-1}), \ \ \forall\, \sigma, \tau \in S_\infty$.
- For every $\sigma \in S_\infty$ have unitary operator $U(\sigma) \in B(\mathcal{H})$ acting by $\big[U(\sigma)\big]\,(\widehat{\tau}) = \widehat{\sigma\tau}, \ \ \forall\, \tau \in S_\infty$.

$\to$ Let $\mathcal{M} := \overline{\mathrm{span}}^{\mathrm{WOT}}\{U(\sigma) \mid \sigma \in S_\infty\} \subseteq B(\mathcal{H})$

(von Neumann algebra generated by the operators $U(\sigma)$).

$\to$ Let $\mathrm{tr} : \mathcal{M} \to \mathbb{C}$ be the vector-state defined by $\widehat{(1)} \in \mathcal{H}$, where $(1) \in S_\infty$ is the identity permutation:

$$\mathrm{tr}(T) := \langle\, T\,\widehat{(1)}\,,\, \widehat{(1)}\,\rangle \text{ for } T \in \mathcal{M}.$$

Standard arguments show that tr is a faithful trace-state on $\mathcal{M}$. We will work with the W^*-probability space $(\mathcal{M}, \mathrm{tr})$.

Law of large numbers for the operators $U(\gamma_n) \in \mathcal{M}$.

Proposition. The sequence $\big(U(\gamma_n) \big)_{n=1}^{\infty}$ has a WOT-limit $A_0 \in \mathcal{M}$, where $A_0 = A_0^*$ and $\| A_0 \| \leq 1$. The operator A_0 can be described via its action on vectors, as follows:

$$\left\{ \begin{array}{c} \text{For every } \sigma, \tau \in S_\infty, \text{ one has } \langle\, A_0(\widehat{\sigma}), \widehat{\tau}\, \rangle = \frac{\mathsf{p}_{1+|V|}}{\mathsf{p}_{|V|}} \langle\, \widehat{\sigma}, \widehat{\tau}\, \rangle, \\ \text{where } V \text{ is the orbit of } \sigma\tau^{-1} \text{ which contains the number 1.} \end{array} \right.$$

Corollary. *(Law of large numbers.)* Let $A_0 \in \mathcal{M}$ be as above. Then

$$\mathrm{SOT} - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} U(\gamma_i) = A_0.$$

A remark about traces of monomials in $U(\gamma_n)$'s and A_0.

There is a nice procedure for computing traces of monomials where every factor either is A_0 or is a $U(\gamma_n)$. Show it on an example – say e.g. we want to compute $\mathrm{tr}\big(U(\gamma_1)A_0^2U(\gamma_2)A_0\big)$.

Trick is: trace will not change when we replace the occurrences of A_0 by "new and distinct" unitaries $U(\gamma_n)$. Can e.g. go with

$$\begin{aligned}
\mathrm{tr}\big(U(\gamma_1)A_0^2U(\gamma_2)A_0\big) &= \mathrm{tr}\big(U(\gamma_1)A_0A_0U(\gamma_2)A_0\big)\\
&= \mathrm{tr}\big(U(\gamma_1)U(\gamma_{10})U(\gamma_{20})U(\gamma_2)U(\gamma_{100})\big)\\
&= \mathrm{tr}\big(U(\gamma_1\gamma_{10}\gamma_{20}\gamma_2\gamma_{100}\big)\\
&= \chi(\gamma_1\gamma_{10}\gamma_{20}\gamma_2\gamma_{100})\\
&= \chi\big(\,(1,\,101,\,3,\,21,\,11,\,2)\,\big)\\
&= \mathsf{p}_6.
\end{aligned}$$

...remark about traces of monomials in $U(\gamma_n)$'s and A_0.

Corollary.

1^o *For every $k \in \mathbb{N}$, one has that $\mathrm{tr}(A_0^k) = \mathsf{p}_{k+1}$.*

2^o *The scalar spectral measure of A_0 with respect to tr is equal to $\sum_{i=1}^d w_i \delta_{w_i}$ (convex combination of Dirac measures).*

3^o *The spectrum of A_0 is equal to $\{t \in (0,1) \mid \exists 1 \leq i \leq d \text{ such that } w_i = t\}$.*

Remark. A_0 is a scalar if and only if we are in the special case of the block character, with $w_1 = \cdots = w_d = 1/d$.

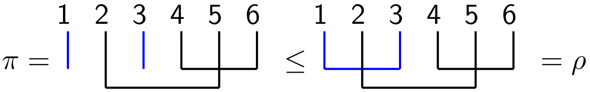
TABLE OF CONTENTS:

Some basic notation concerning set-partitions.

$\mathcal{P}(k)$ denotes the set of all partitions of $\{1, \ldots, k\}$.

On $\mathcal{P}(k)$ have partial order given by reverse refinement: "$\pi \leq \rho$" means that every block of π is contained in some block of ρ.

An example of $\pi \leq \rho$ in $\mathcal{P}(6)$:



Some special subsets of $\mathcal{P}(k)$: we let

$$\mathcal{P}_2(k) := \{\pi \in \mathcal{P}(k) \mid \text{every block } V \in \pi \text{ has } |V| = 2\},$$

$$\mathcal{P}_{\leq 2}(k) := \{\pi \in \mathcal{P}(k) \mid \text{every block } V \in \pi \text{ has } |V| \leq 2\}.$$

For instance, the example shown above has $\pi \in \mathcal{P}_{\leq 2}(6)$ and $\rho \in \mathcal{P}_2(6)$.

Review of "exchangeable" CLT.

Notation. We will treat a tuple $\underline{i} \in \mathbb{N}^k$ as a function $\underline{i} : \{1, \ldots, k\} \to \mathbb{N}$. The *kernel* of such an $\underline{i}$ is the partition $\mathrm{Ker}(\pi) \in \mathcal{P}(k)$ defined as follows: two numbers $p, q \in \{1, \ldots, k\}$ belong to the same block of $\mathrm{Ker}(\underline{i})$ if and only if $\underline{i}(p) = \underline{i}(q)$.

Definition *(Exchangeable sequence).* $(\mathcal{A}, \varphi)$ $*$-probability space and $(a_n)_{n=1}^{\infty}$ selfadjoint elements of $\mathcal{A}$. Quantities

$$\varphi(a_{\underline{i}(1)} \cdots a_{\underline{i}(k)}), \text{ with } k \in \mathbb{N} \text{ and } \underline{i} : \{1, \ldots, k\} \to \mathbb{N}$$

are called *joint moments* of $(a_n)_{n=1}^{\infty}$. We say that $(a_n)_{n=1}^{\infty}$ is *exchangeable* to mean that it satisfies

$$\begin{cases} \varphi(a_{\underline{i}(1)} \cdots a_{\underline{i}(k)}) = \varphi(a_{\underline{j}(1)} \cdots a_{\underline{j}(k)}) \\ \text{for every } k \in \mathbb{N} \text{ and } \underline{i}, \underline{j} \in \mathbb{N}^k \text{ such that } \mathrm{Ker}(\underline{i}) = \mathrm{Ker}(\underline{j}). \end{cases}$$

...review of "exchangeable" CLT...

Definition. $(\mathcal{A}, \varphi)$ $*$-probability space and let $(a_n)_{n=1}^{\infty}$ be an exchangeable sequence of selfadjoint elements of $\mathcal{A}$.

1^o Have a *function on partitions* $\mathbf{t} : \sqcup_{k=1}^{\infty} \mathcal{P}(k) \to \mathbb{C}$ associated to $(a_n)_{n=1}^{\infty}$, where for $k \in \mathbb{N}$ and $\pi \in \mathcal{P}(k)$ we put

$$\begin{cases} \mathbf{t}(\pi) := \varphi\big(a_{\underline{i}(1)} \cdots a_{\underline{i}(k)}\big), \text{ where } \underline{i} \in \mathbb{N}^k \text{ is} \\ \text{any } k\text{-tuple such that } \ker(\underline{i}) = \pi. \end{cases}$$

This formula is unambiguous due to exchangeability.

2^o $(a_n)_{n=1}^{\infty}$ is said to have the *singleton vanishing property* when its function on partitions $\mathbf{t}$ satisfies:

$$\begin{cases} \mathbf{t}(\pi) = 0 \text{ whenever the partition } \pi \in \sqcup_{k=1}^{\infty} \mathcal{P}(k) \\ \text{has at least one block } V \text{ with } |V| = 1. \end{cases}$$

Remark. Singleton vanishing property guarantees centering (use the unique partition in $\mathcal{P}(1)$ to get $\varphi(a_n) = 0$ for all n).

...review of "exchangeable" CLT.

Theorem (Bozejko-Speicher 1996). *Let* $(\mathcal{A}, \varphi)$ *be a* $*$*-probability space and let* $(a_n)_{n=1}^{\infty}$ *be a sequence of selfadjoint elements of* $\mathcal{A}$ *which is exchangeable and has the singleton vanishing property. Let* $\mathbf{t} : \sqcup_{k=1}^{\infty} \mathcal{P}(k) \to \mathbb{C}$ *be the function on partitions associated to* $(a_n)_{n=1}^{\infty}$*. Consider the linear functional* $\mu : \mathbb{C}[X] \to \mathbb{C}$ *defined by asking that* $\mu(1) = 1$ *and that*

$$\mu(X^k) = \sum_{\rho \in \mathcal{P}_2(k)} \mathbf{t}(\rho), \quad \forall\, k \in \mathbb{N}$$

(with right-hand side equal to 0 *for* k *odd). Then:*

1^o μ *is positive (that is,* $\mu(\, P \cdot \overline{P}\,) \geq 0$ *for every* $P \in \mathbb{C}[X]$*).*

2^o *For every* $n \in \mathbb{N}$ *put* $s_n := \frac{1}{\sqrt{n}}\big(a_1 + \cdots + a_n\big) \in \mathcal{A}$*. Then* $(s_n)_{n=1}^{\infty}$ *converges in moments to* μ*, that is, one has* $\lim_{n\to\infty} \varphi(s_n^k) = \mu(X^k)$ *for every* $k \in \mathbb{N}$*.*

Exchangeable CLT for the sequence of $U(\gamma_n)$'s.

Go back to our framework of $(\mathcal{M}, \mathrm{tr})$. Recall that:

$$\mathcal{M} := \overline{\mathrm{span}}^{\mathrm{WOT}}\{U(\sigma) \mid \sigma \in S_\infty\} \subseteq B(\mathcal{H}),$$

$$\mathrm{tr}(T) := \langle\, T\, \widehat{(1)}\,,\, \widehat{(1)}\, \rangle \text{ for } T \in \mathcal{M},$$

$$A_0 = \text{SOT-lim}_{n\to\infty}\Big(\, \tfrac{1}{n}\sum_{i=1}^n U(\gamma_i)\,\Big) \in \mathcal{M}.$$

Remark. The operators $U(\gamma_n)$ are not centered, they have

$$\mathrm{tr}\big(\, U(\gamma_n)\,\big) = \chi(\gamma_n) = \mathsf{p}_2, \quad \forall\, n \in \mathbb{N}.$$

In preparation of a CLT result, we now want to center the $U(\gamma_n)$'s.

Important point: the way to do the centering is by subtracting the limit provided by the law of large numbers! That is, we go like this:

Notation. $\overset{\diamond}{U}_n := U(\gamma_n) - A_0 \in \mathcal{M}, \quad n \in \mathbb{N}.$

...exchangeable CLT for the sequence of $U(\gamma_n)$'s...

Notation. $\overset{\diamond}{U}_n := U(\gamma_n) - A_0 \in \mathcal{M}, \quad n \in \mathbb{N}.$

Remark. The elements $\overset{\diamond}{U}_n$ are indeed centred:

$$\mathrm{tr}(\overset{\diamond}{U}_n) = \mathrm{tr}\big(U(\gamma_n) \big) - \mathrm{tr}(A_0) = \mathsf{p}_2 - \mathsf{p}_2 = 0, \quad n \in \mathbb{N}.$$

Remark. Typically, the centering procedure goes by subtracting a scalar, and would yield elements $\overset{\circ}{U}_n := U(\gamma_n) - \mathsf{p}_2$, $n \in \mathbb{N}$. The notation "$\overset{\diamond}{U}_n$" goes in the same spirit. But, unless we are in the special case of a block character (with $w_1 = \cdots = w_d = 1/d$), we have $\overset{\diamond}{U}_n \neq \overset{\circ}{U}_n$. This distinction is important, as the proposition shown next (more precisely: verifying the singleton-vanishing property) would not work in connection to the $\overset{\circ}{U}_n$'s.

...exchangeable CLT for the sequence of $U(\gamma_n)$'s.

Proposition. *The sequence* $\big(\overset{\diamond}{U}_n\big)_{n=1}^{\infty}$ *in* $(\mathcal{M}, \mathrm{tr})$ *is exchangeable and has the singleton-vanishing property.*

Feed this into the exchangeable CLT theorem, to get:

Corollary (CLT for the $U(\gamma_n)$'s). *Let* $\mathbf{t}$ *be the function on partitions associated to* $(\overset{\diamond}{U}_n)_{n=1}^{\infty}$ *and let* $\mu_{\underline{w}} : \mathbb{C}[X] \to \mathbb{C}$ *be the linear functional defined by asking that* $\mu(1) = 1$ *and that*

$$\mu_{\underline{w}}(X^k) = \sum_{\rho \in \mathcal{P}_2(k)} \mathbf{t}(\rho), \quad k \in \mathbb{N}.$$

Then $\mu_{\underline{w}}$ *is positive and the sequence of elements*

$$s_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \overset{\diamond}{U}_i \; = \; \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (U(\gamma_i) - A_0) \in \mathcal{M}, \quad n \geq 1,$$

converge to $\mu_{\underline{w}}$ *in moments.*

A bit of combinatorial detail concerning $\mathbf{t}(\rho)$.

We obtained a limit law $\mu_{\underline{w}}$ with moments

$$\mu_{\underline{w}}(X^k) = \sum_{\rho \in \mathcal{P}_2(k)} \mathbf{t}(\rho), \quad k \in \mathbb{N},$$

where $\mathbf{t}$ is the function on partitions associated to the exchangeable sequence $(\overset{\diamond}{U}_n)_{n=1}^{\infty}$. In order to understand what is $\mu_{\underline{w}}$, we need a good handle on $\mathbf{t}$. Useful formula: for $k \in \mathbb{N}$ even and $\rho \in \mathcal{P}_2(k)$ one has

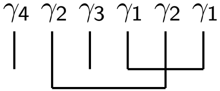
$$\mathbf{t}(\rho) = \sum_{\substack{\pi \in \mathcal{P}_{\leq 2}(k) \\ \pi \leq \rho}} (-1)^{|\pi|_1/2} \chi(\tau_\pi),$$

where $\tau_\pi \in S_\infty$ is a product of star-transpositions canonically associated to π, and $|\pi|_1$ is the number of singleton-blocks of π.

...a bit of combinatorial detail concerning $\mathbf{t}(\rho)$.

...Formula for $\mathbf{t}(\rho)$ uses "$\chi(\tau_\pi)$" where $\tau_\pi \in S_\infty$ is a product of star-transpositions canonically associated to π. For instance for

$\pi = \Big\{ \{1\}, \{2,5\}, \{3\}, \{4,6\} \Big\}$ we draw



Hence $\tau_\pi = \gamma_4\gamma_2\gamma_3\gamma_1\gamma_2\gamma_1 = (1,5)(1,3)(1,4)(1,2)(1,3)(1,2)$
$= (1,4,3,2,5)$, with $\chi(\tau_\pi) = p_5$.

Remark. By starting from the same character χ of S_∞, one can consider a construction of a positive function $\mathbf{v}$ on pair-partitions introduced by Bozejko-Guta in 2002. There they have

$$\mathbf{v}(\rho) := \chi\big(\,\theta_\rho\,\big), \quad \text{for } \rho \in \mathcal{P}_2(k),$$

where θ_ρ is another permutation canonically associated to ρ.
(Relation between τ_ρ and θ_ρ...(?))

TABLE OF CONTENTS:

Recap: $d \times d$ GUE matrix – how to build it. Suppose that:

$\rightarrow$ For every $1 \leq i < j \leq d$ we have a centred complex Gaussian random variable $g_{i,j}$ of variance $1/d$.

$\rightarrow$ We have an independent family of centred real Gaussian random variables $g_{1,1}, \ldots, g_{d,d}$, where every $g_{i,i}$ has variance $1/d$.

$\rightarrow$ The $\frac{d(d-1)}{2} + 1$ Gaussian families $\{\text{Re}(g_{1,2}), \text{Im}(g_{1,2})\}, \ldots,$ $\{\text{Re}(g_{d-1,d}), \text{Im}(g_{d-1,d})\}$, $\{g_{1,1}, \ldots, g_{d,d}\}$ are independent.

Then for every $1 \leq i < j \leq d$ put $g_{j,i} := \overline{g_{i,j}}$, and form the random matrix $G = [g_{i,j}]_{i,j=1}^{d}$. This is our desired GUE matrix.
(The entries of G are scaled such that the expected normalized trace of G^2 is $E(\mathrm{tr}_d(G^2)) = 1$.)

Traceless $d \times d$ GUE matrix.

Definition. Let $G = [g_{i,j}]_{i,j=1}^d$ be a GUE matrix with entries scaled such that the expected normalized trace of G^2 is $E(\mathrm{tr}_d(G^2)) = 1$. The *traceless GUE* is the random matrix M obtained from this G by projecting the random vector $(g_{1,1}, \ldots, g_{d,d}) \in \mathbb{R}^d$ onto the hyperplane of equation $t_1 + \cdots + t_d = 0$. Thus

$$M := G - \frac{g_{1,1} + \cdots + g_{d,d}}{d} I_d,$$

where I_d is the identity $d \times d$ matrix.

Remark. The diagonal entries of M are linearly dependent. They form a Gaussian family of centred random variables with covariance matrix $C = [c_{i,j}]_{i,j=1}^d$, where:

$$c_{i,i} = 1/d - 1/d^2, \ \ \forall\, 1 \le i \le d,$$
$$c_{i,j} = c_{j,i} = -1/d^2, \ \ \forall\, 1 \le i < j \le d.$$

Traceless $d \times d$ GUE matrix – how to build it. Suppose that:

- $\rightarrow$ For every $1 \le i < j \le d$ we have a centred complex Gaussian random variable $f_{i,j}$ of variance $1/d$.
- $\rightarrow$ We have a Gaussian family of centred real random variables $f_{1,1}, \ldots, f_{d,d}$, with covariance matrix $C = [c_{i,j}]_{i,j=1}^{d}$ as shown above ($c_{i,i} = 1/d - 1/d^2$ and $c_{i,j} = c_{j,i} = -1/d^2$ for $i \neq j$).
- $\rightarrow$ The $\frac{d(d-1)}{2} + 1$ Gaussian families $\{\mathrm{Re}(f_{1,2}), \mathrm{Im}(f_{1,2})\}, \ldots,$ $\{\mathrm{Re}(f_{d-1,d}), \mathrm{Im}(f_{d-1,d})\}$, $\{f_{1,1}, \ldots, f_{d,d}\}$ are independent.

Then for every $1 \le i < j \le d$ put $f_{j,i} := \overline{f_{i,j}}$, and form the random matrix $M = [f_{i,j}]_{i,j=1}^{d}$. This is our desired traceless GUE matrix.

In order to get a CCR-analogue of this, we will use a CCR analogue for the notion of centred complex Gaussian random variable.

CCR analogue for a centred complex Gaussian r.v.

Definition. Let $(\mathcal{A}, \varphi)$ and $\omega_{(1,*)}, \omega_{(*,1)} \in (0, \infty)$ be given. Say that an $a \in \mathcal{A}$ is a *centred CCR-complex-Gaussian element* with variances $\omega_{(1,*)}$ and $\omega_{(*,1)}$ to mean that we have the relation

$$(*) \qquad a^* a - \omega(*,1)\, 1_{\mathcal{A}} = a a^* - \omega_{(1,*)}\, 1_{\mathcal{A}},$$

and that for $p, q \in \mathbb{N} \cup \{0\}$ we have expectations

$$(**) \quad \varphi\big(a^p (a^*)^p \big) = p!\, \omega_{(1,*)}^p \text{ and } \varphi\big(a^p (a^*)^q \big) = 0 \text{ for } p \neq q.$$

Remark. Usual "complex Gaussian" is retrieved when we set $\omega_{(1,*)} = \omega_{(*,1)} =: \omega$. Then (*) says that a commutes with a^* (hence can be treated like a usual complex r.v.) while (**) becomes the formula giving the joint moments of f and $\overline{f}$ for a centred complex Gaussian variable f of variance ω.

Wick's Lemma for a centred CCR-complex-Gaussian.

Recall the two relations used in the definition:

$$(*) \qquad a^* a = a a^* + \big(\omega_{(*,1)} - \omega_{(1,*)}\big)\, 1_{\mathcal{A}}, \qquad \text{and}$$

$$(**) \qquad \varphi\big(\, a^p (a^*)^q \,\big) = \delta_{p,q} \cdot p!\, \omega_{(1,*)}^p .$$

These relations determine all the joint moments of a and a^*. One has a CCR version of Wick's Lemma which computes such joint moments. Namely, for $k \in \mathbb{N}$ and $\varepsilon(1), \ldots, \varepsilon(k) \in \{1, *\}$, one has

$$\varphi\big(a^{\varepsilon(1)} \cdots a^{\varepsilon(k)}\big) = \sum_{\rho \in \mathcal{P}_2(k)} \Big[\prod_{\substack{\{p,q\} \in \rho \\ \text{with } p<q}} \omega_{(\varepsilon(p), \varepsilon(q))} \Big],$$

where we make the convention to put $\omega_{(1,1)} = \omega_{(*,*)} := 0$.

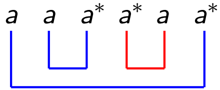
...Wick's Lemma for a centred CCR-complex-Gaussian.

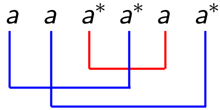
...CCR version of Wick's Lemma:

$$\varphi\left(a^{\varepsilon(1)} \cdots a^{\varepsilon(k)}\right) = \sum_{\rho \in \mathcal{P}_2(k)} \left[\prod_{\substack{\{p,q\} \in \rho, \\ \text{with } p<q}} \omega_{(\varepsilon(p),\varepsilon(q))} \right]$$

(where $\omega_{(1,*)}, \omega_{(*,1)}$ are variances and $\omega_{(1,1)} = \omega_{(*,*)} := 0$).

For example $\varphi\left(a\, a\, a^*\, a^*\, a\, a^*\right) = 2\omega_{(1,*)}^3 + 4\omega_{(1,*)}^2\, \omega_{(*,1)}$. Have 6 terms, e.g. two of the terms $\omega_{(1,*)}^2 \cdot \omega_{(*,1)}$ come from

$\rho_1 =$ 

and $\rho_2 =$ 

Note: Wick's Lemma confirms that, symmetric to (**), one has

$$(***) \qquad \varphi\left(\,(a^*)^q\, a^p\,\right) = \delta_{p,q} \cdot q!\,\omega_{(*,1)}^q, \quad \text{for } p, q \in \mathbb{N} \cup \{0\}.$$

CCR analogue for a traceless $d \times d$ GUE matrix.

We also input our weights $w_1 \geq \cdots \geq w_d > 0$ with $\sum_{i=1}^{d} w_i = 1$. Suppose we have a $*$-probability space $(\mathcal{A}, \varphi)$ and a family $\{\mathcal{A}_o\} \cup \{\mathcal{A}_{i,j} \mid 1 \leq i < j \leq d\}$ of unital $*$-subalgebras of $\mathcal{A}$ which are commuting independent. Suppose moreover that:

(i) For every $1 \leq i < j \leq d$, we have an element $a_{i,j} \in \mathcal{A}_{i,j}$ which is centred CCR-complex-Gaussian with parameters w_j and w_i. Put $a_{j,i} := a_{i,j}^* \in \mathcal{A}_{i,j}$ (thus have $a_{j,i}a_{i,j} = a_{i,j}a_{j,i} + (w_i - w_j)\, 1_{\mathcal{A}}$).

(ii) $\mathcal{A}_o$ is commutative and we have selfadjoint $a_{1,1}, \ldots, a_{d,d} \in \mathcal{A}_o$ which form a centred Gaussian family with covariance matrix $C = [c_{i,j}]_{i,j=1}^{d}$, where:

$$\begin{aligned} c_{i,i} &= w_i - w_i^2 && \text{for } 1 \leq i \leq d; \\ c_{i,j} &= c_{j,i} = -w_i w_j && \text{for } 1 \leq i < j \leq d. \end{aligned}$$

Then the selfadjoint matrix $M = [a_{i,j}]_{1 \leq i,j \leq d}$ in $M_d(\mathcal{A})$ is said to be a *traceless CCR-GUE matrix with parameters* $w_1, \ldots, w_d$.

TABLE OF CONTENTS:

Main result of arXiv:2203.01763 (Campbell-Köstler-Nica).

Theorem. *Let* $(\mathcal{A}, \varphi)$ *be a* $*$*-probability space, and let* $M = [a_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{A})$ *be a traceless CCR-GUE matrix with parameters* $w_1, \ldots, w_d$ *(as in discussion from Part III). Consider the linear functional* $\varphi_{\underline{w}} : M_d(\mathcal{A}) \to \mathbb{C}$ *defined by*

$$\varphi_{\underline{w}}(X) = \sum_{i=1}^{d} w_i \, \varphi(x_{i,i}), \quad \text{for } X = [x_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{A}).$$

Then the law of M *in the* $*$*-probability space* $(M_d(\mathcal{A}), \varphi_{\underline{w}})$ *is equal to the limit law* $\mu_{\underline{w}}$ *from the theorem of CLT type discussed in Part II of the talk.*

Line of proof of the main result.

The result is that the law of the traceless CCR-GUE matrix M, in the $*$-probability space $(M_d(\mathcal{A}), \varphi_{\underline{w}})$, is equal to the limit law $\mu_{\underline{w}}$ from Part II of the talk.
For the proof, it suffices to establish equalities of even moments,

$$(\diamondsuit) \qquad \varphi_{\underline{w}}(M^k) = \int_{\mathbb{R}} t^k \, d\mu_{\underline{w}}(t), \text{ for all even } k \in \mathbb{N}.$$

This is because the odd moments vanish, while for even moments it is easy to give estimates showing that they give a uniquely determined probability distribution.

In order to establish $(\diamondsuit)$, we prove that both its sides are described by the same Wick-style formula.

...line of proof of the main result.

...suffices to prove:

$$(\diamondsuit) \qquad \varphi_{\underline{w}}(M^k) = \int_{\mathbb{R}} t^k \, d\mu_{\underline{w}}(t), \text{ for all even } k \in \mathbb{N}.$$

In order to establish $(\diamondsuit)$, we prove that both its sides are described by the same Wick-style formula.

On LHS, the Wick-style formula is obtained by putting together the Wick formulas which we know to hold for the entries of $M = [a_{i,j}]_{i,j=1}^d$. (Usual Wick for $\{a_{1,1}, \ldots, a_{d,d}\}$ and CCR-Wick for every $\{a_{i,j}, a_{j,i}\}$ with $1 \leq i < j \leq d$.)

On RHS, start from the formula offered by the exchangeable CLT:

$$\int_{\mathbb{R}} t^k \, d\mu_{\underline{w}}(t) = \sum_{\rho \in \mathcal{P}_2(k)} \mathbf{t}(\rho) \ = \ \cdots$$

Use combinatorics of partitions to process this, until a Wick-style formula emerges.

Case of the block-character, and the $d \to \infty$ limit.

Assume $w_1 = \cdots = w_d = 1/d$, and write "μ_d" instead of $\mu_{\underline{w}}$.
Above theorem gives μ_d as law of a usual traceless GUE matrix M:

$$(\square) \quad M = G - \frac{g_{1,1} + \cdots + g_{d,d}}{d} I_d, \quad \text{where } G \text{ is GUE.}$$

From $(\square)$ it is easy to infer that $\mu_d * N(0, 1/d^2) = \nu_d$, where:
$N(0, 1/d^2) =$ centred normal distribution of variance $1/d^2$, and
$\nu_d =$ law (a.k.a. "average empirical eigenvalue distribution") of G.

Finally, make $d \to \infty$. Have fundamental fact that ν_d converges to the semicircle law – hence so does μ_d. On the other hand, the block-characters converge to the canonical trace of S_∞, since $\lim_{d\to\infty}(1/d)^{||\sigma||} = 0$ for every $\sigma \neq (1)$ in S_∞.

In this way, the $d \to \infty$ limit can be invoked to retrieve the theorem of Biane reviewed at the beginning of the talk.

Thank you for your attention!