Western UNIVERSITY · CANADA

COVARIANT AND CONTRAVARIANT FUNCTORIALITY FOR OPERATOR ALGEBRAS THAT ONE CAN SEE

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Joint work with Mariusz Tobolski.

London, Ontario, 25 May 2023

Morphisms of graphs

Definition

A homomorphism $f\colon E \to F$ of graphs is a pair of maps

$$(f^0 : E^0 \to F^0, f^1 : E^1 \to F^1)$$

satisfying the conditions:

$$s_F \circ f^1 = f^0 \circ s_E\,, \qquad t_F \circ f^1 = f^0 \circ t_E\,.$$

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Definition

A path homomorphism of graphs is a map $f\colon FP(E) \to FP(F)$ satisfying:

1. $f(E^0) \subseteq F^0$,
2. $s_F \circ f = f \circ s_E\,, \quad t_F \circ f = f \circ t_E\,,$
3. $\forall\, p, q \in FP(E)$ such that $t(p) = s(q)$: $f(pq) = f(p)f(q)$.

Note that a path homomorphism of graphs is a homomorphism of graphs if and only if it preserves the lengths of paths.

Leavitt path algebras and graph C*-algebras

Definition

Let E be a graph and k be a field. The Leavitt path algebra $L_k(E)$ of E is the path algebra $k\bar{E}$ of the extended graph $\bar{E}$ divided by the ideal generated by the union of the following sets:

1. $\{\chi_{e^*}\chi_f - \delta_{e,f}\chi_{t(e)} \mid e, f \in E^1\}$,
2. $\{\sum_{e \in s^{-1}(v)} \chi_e \chi_{e^*} - \chi_v \mid v \in \mathrm{reg}(E)\}$.

Here $\mathrm{reg}(E)$ is the set of all regular vertices of E, and a vertex is called regular iff it emits at l east one edge and at most finitely many edges.

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Definition

Let E be a graph and $k = \mathbb{C}$ be a field. Then the formulas

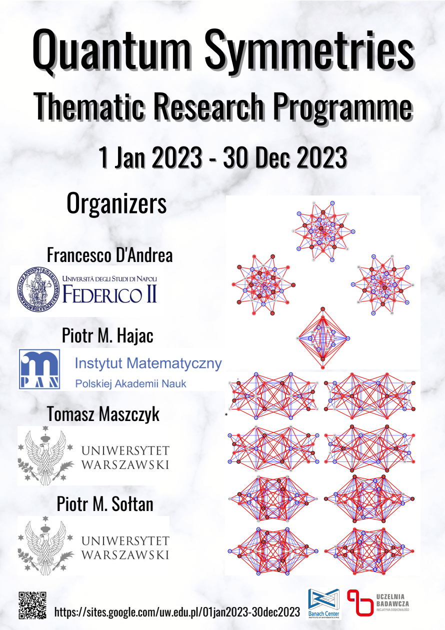
$$\forall\, v \in E^0 \colon (\chi_v)^* := \chi_v\,,\ \forall\, e \in E^1 \colon (\chi_e)^* := \chi_{e^*}\,,\ (\chi_{e^*})^* := \chi_e\,,$$

define involutions rendering $\mathbb{C}\bar{E}$, $C_{\mathbb{C}}(E)$, and $L_{\mathbb{C}}(E)$ $*$-algebras. The universal C*-algebra of $L_{\mathbb{C}}(E)$ is called the graph C*-algebra of E, and denoted by $C^*(E)$.

Staff Exchange network of 78 mathematicians from 26 places:

IMPAN (Poland), University of Warsaw (Poland), University of Wrocław (Poland), Jagiellonian University (Poland), University of Copenhagen (Denmark), University of Southern Denmark, Odense (Denmark), SISSA, Trieste (Italy), University of Naples Federico II (Italy), Leiden University (Netherlands), University of Göttingen (Germany), University of Haifa (Israel), University of Hawai'i at Mānoa (USA), University of California at Berkeley (USA), Pomona College (USA), Arizona State University (USA), University of Denver (USA), University of Colorado at Boulder (USA), University of Colorado at Colorado Springs (USA), University of Kansas at Lawrence (USA), Kansas State University (USA), Penn State University (USA), State University of New York at Buffalo (USA), Fields Institute (Canada), Western University (Canada) University of New Brunswick at Fredericton (Canada), University Michoacana de San Nicolás de Hidalgo (Mexico).

Quantum Symmetries 2023



Main activities:

- Simons Semester
  Operator Algebras
  That One Can See
- Banach Center
  Conference
  Graph Algebras
- North Atlantic
  Noncommutative
  Geometry Seminar

Contravariant conditions

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2. $TBPOG$ is the subcategory of POG whose morphisms $f\colon E \to F$ satisfy the target-bijectivity condition:

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3. $CRTBPOG$ is the subcategory of $TBPOG$ whose morphisms $f\colon E \to F$ satisfy

$$f\big(E^0 \setminus \mathrm{reg}(E)\big) \subseteq F^0 \setminus \mathrm{reg}(F).$$

Main contravariant result

Let KA denote the category of algebras over k and algebra homomorphisms, and UKA denote its subcategory of unital algebras and unital algebra homomorphisms.

Theorem

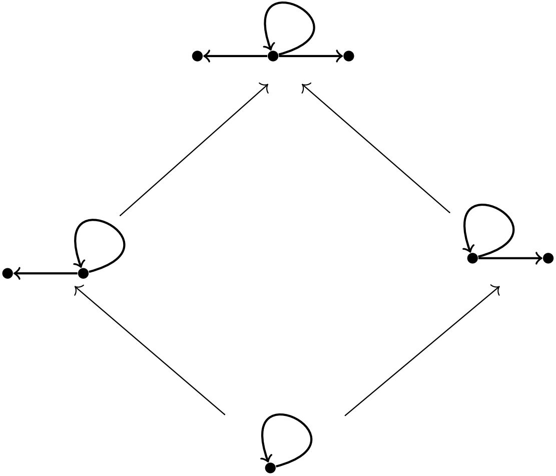
The assignment

$$\mathrm{Obj}(\mathrm{CRTBPOG}) \ni E \longmapsto L_k(E) \in \mathrm{Obj}(KA),$$
$$\mathrm{Mor}(\mathrm{CRTBPOG}) \ni ((f^0, f^1)\colon E \to F) \longmapsto$$
$$\big(\bar{f}^*\colon L_k(F) \to L_k(E)\big) \in \mathrm{Mor}(KA),$$
$$L_k(F) \ni [\chi_p] \overset{\bar{f}^*}{\longmapsto} \sum_{q \in \bar{f}^{-1}(p)} [\chi_q] \in L_k(E)\,,$$

where $\bar{f}\colon FP(\bar{E}) \to FP(\bar{F})$ *is the map induced by* (f^0, f^1)*,*

defines a contravariant functor. Furthermore, the same assignment restricted to the subcategory given by graphs with finitely many vertices yields a contravariant functor to the category UKA*.*

Equatorial Podleś quantum sphere



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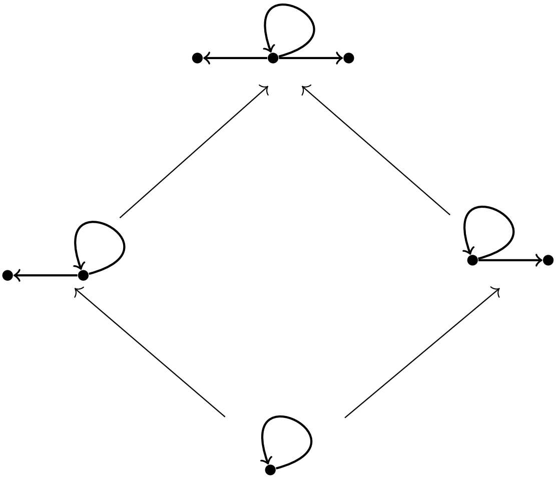

Theorem

A pushout-to-pullback theorem for Leavitt path algebras and graph C-algebras generalizing previous such a theorem for unions of graphs by Reznikoff, Hajac and Tobolski.*

Covariant conditions

C1 IPG is the category of graphs and path homomorphisms of graphs that are injective when restricted to vertices.

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when restricted to the sets of edges.

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C3 $RMIPG$ is the subcategory of $MIPG$ whose morphisms satisfy the regularity conditions:

(A) For any $v \in \mathrm{reg}(E)$, the vertex $f(v)$ emits $|s_E^{-1}(v)|$-many positive-length paths $p_1, \ldots, p_{n_v}$, $n_v := |s_E^{-1}(v)|$, whose all edges begin at regular vertices. Also, we require that the set $FP_{f(v)} := \{p_1, \ldots, p_{n_v}\}$ is constructed in the following way: we take $x \in s_F^{-1}(f(v))$ and either set it aside as a length-one element of $FP_{f(v)}$, or extend it by all edges emitted from $t_F(x)$. Any thus obtained path of length two, we either set aside as an element of $FP_{f(v)}$, or extend it by all edges emitted from its end. Then we iterate this procedure until we obtain the set $FP_{f(v)}$.

(B) For any $v \in \mathrm{reg}(E)$, the map f when restricted to $s_E^{-1}(v)$ is a bijection onto $FP_{f(v)}$.

Main covariant result

Theorem

The directed graphs together with path homomorphisms satisfying the three covariant conditions form a subcategory in the category of graphs and path homomorphisms. We call this subcategory RMIPG. Moreover, the assignments

$$\forall\ E \in \mathrm{Obj}(RMIPG)\colon E \overset{f_*^L}{\longmapsto} C^*(E),$$

$$\forall\ (f\colon E \to F) \in \mathrm{Mor}(RMIPG),\ p \in FP(\bar{E})\colon$$

$$C^*(E) \ni [\chi_p] \overset{f_*^L}{\longmapsto} [\chi_{\bar{f}(p)}] \in C^*(F),$$

define a covariant functor into the category of C-algebras and *-homomorphisms.*

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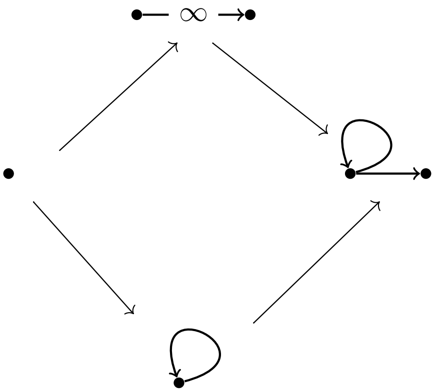
$$C^*(E) \ni [\chi_p] \overset{f_*^L}{\longmapsto} [\chi_{\bar{f}(p)}] \in C^*(F),$$

define a covariant functor into the category of C-algebras and *-homomorphisms. Finally, restricting the functor to the subcategory $RMBPG$ (graphs have finitely many vertices and path homomorphisms are bijective when restricted to the sets of vertices) gives a functor into the subcategory of unital C*-algebras and unital *-homomorphisms.*

Quantum $\mathbb{C}P^1$



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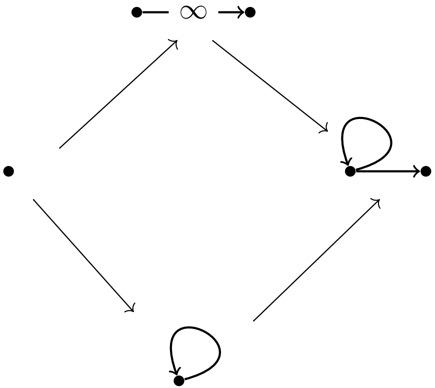
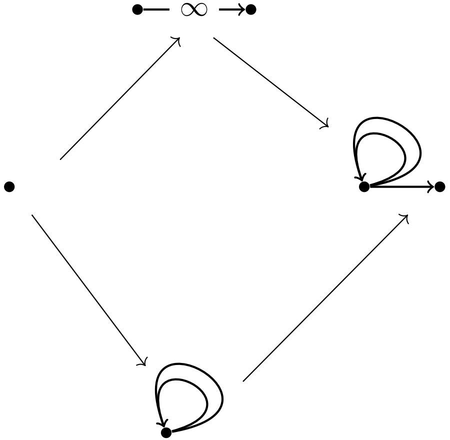


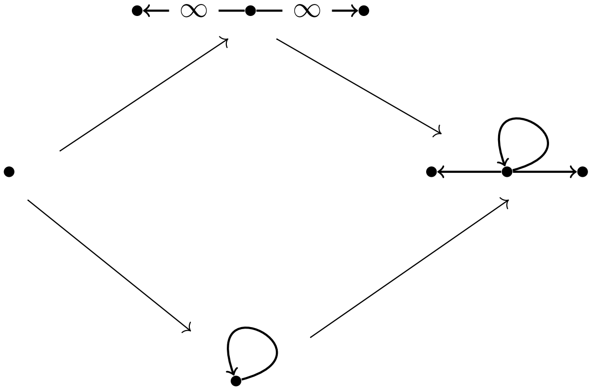
Theorem (A. Chirvasitu, P.M.H., M. Tobolski)

A pullback theorem for graph C-algebras. Currently working on its generalization using the main covariant theorem.*

A quantum bonus



Quantum weighted complex projective line



Quantum $\mathbb{C}P^2$