Inductive sequences of spectral triples for twisted group C^*-algebras and convergence in the spectral propinquity

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“Distances" between C^*-algebras

Physicists often use finite dimensional matrix algebras to approximate objects they study. To have matrix algebras ‘converge’ to a specific unital C^*-algebra in the sense of **inductive limits**, the resulting C^*-algebra will be an AF-algebra. Therefore Rieffel, and subsequently Latrémolière, developed a different “distance", Latrémolière’s (dual) Gromov-Hausdorff propinquity, between unital C^*-algebras. This has the benefit that more general unital C^*-algebras can be approximated by matrix algebras. When restricted to unital commutative C^*-algebras, this propinquity distance is comparable to the Edwards-Gromov-Hausdorff distance between compact metric spaces.

More recently, Latrémolière has extended this definition to define the spectral propinquity between spectral triples (based on triples of unital C^*-algebras, representations on Hilbert spaces, Dirac operators).

“Distances" via length functions

Here, we put a compact metric space structure on twisted group C^*-algebras corresponding to some discrete abelian groups that are **not** finitely generated and have an inductive limit structure.

We do this by constructing even metric spectral triples on these unital C^*-algebras, and then forming the corresponding seminorms from the Dirac operators coming from length functions with special properties. We also show that our spectral triples are limits in Latrémoliére's spectral propinquity of spectral triples on noncommutative tori, and that the whose corresponding quantum compact metric spaces converge in the dual Gromov-Hausdorff propinquity to the limit quantum metric space.

We start by giving some background for the Rieffel–Latrémolière concepts.

Distance between metric spaces transformed: towards 'propinquity'

If X and Y are two compact metric spaces, then the Gromov-Hausdorff distance $\mathrm{d}_{GH}(X,Y)$ is defined to be the infimum of all Hausdorff distances $\mathrm{d}_H(f(X),g(Y))$ for all metric spaces M and all isometric embeddings $f: X \to M$ and $g: Y \to M$.

Latrémolière's (dual) Gromov-Hausdorff propinquity Λ^* defined on (separable) unital C^*-algebras/seminorm pairs, will have the property that for X and Y compact metric spaces,

$$\Lambda^*((C(X),L_X),(C(Y),L_Y) \leq \mathrm{d}_{GH}(X,Y).$$

Here, for $f \in C(X)$, $\mathbb{L}_X(f) = \sup_{x_1 \neq x_2 \in X} \left\{ \frac{|f(x_1)-f(x_2)|}{d_X(x_1,x_2)} \right\}$, with L_Y similarly defined.

(Leibniz) Compact quantum metric spaces

Definition 1 [Rie], [La] A **(Leibniz) compact quantum metric space** $(\mathcal{A}, L)$ is an ordered pair, where $\mathcal{A}$ is a unital C^*-algebra and L is a *lower semi-cont's Lipschitz* seminorm defined on a dense subspace over $\mathbb{R}$, $\operatorname{dom} L$ of the self-adjoint elements of $\mathcal{A}$, $sa(\mathcal{A})$, such that:

(1) $\{a \in sa(\mathcal{A}) : L(a) = 0\} = \mathbb{R} \cdot 1_{\mathcal{A}}$,

(2) the *Monge-Kantorovich* metric mk_L defined on the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ by setting for all $\varphi, \psi \in \mathcal{S}(\mathcal{A})$,
$$\mathrm{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(L), L(a) \leq 1\},$$
metrizes the weak-* topology restricted to the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$.

(3) The seminorm L satisfies the **Leibniz inequality of Jordan-Lie type**:
$$\max\left\{L\left(\frac{ab+ba}{2}\right), L\left(\frac{ab-ba}{2i}\right)\right\} \leq L(a)\,||b||_{\mathcal{A}} + ||a||_{\mathcal{A}}\,L(b).$$

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Compact quantum metric spaces and quantum isometries

Definition 2 Let $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ be two compact quantum metric spaces. A quantum isometry $\Theta : (\mathcal{A}, L_{\mathcal{A}}) \to (\mathcal{B}, L_{\mathcal{B}})$ is a unital $*$-homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ such that $L_{\mathcal{B}}$ is the quotient seminorm on $\mathcal{A}$ induced by Θ, i.e.

$$\forall b \in \operatorname{dom} L_{\mathcal{B}},\ L_{\mathcal{B}}(b) \ = \ \inf\{L_{\mathcal{A}}(a) :\ a \in \ \operatorname{dom} L_{\mathcal{A}},\ \Theta(a) = b\}.$$

If, in addition, Θ is a $*$-automorphism from $\mathcal{A}$ onto $\mathcal{B}$ and $L_{\mathcal{B}} = L_{\mathcal{A}} \circ \Theta$, and in addition Θ^{-1} is also a quantum isometry from $(\mathcal{B}, L_{\mathcal{B}})$ and $(\mathcal{A}, L_{\mathcal{A}})$, we say that Θ is a full quantum isometry.

Remark: Latrémolière has shown that if Θ is a full quantum isometry between the compact quantum metric spaces $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ then the dual map of Θ, $\phi \ \to \phi \circ \Theta$ is an isometry of $(\mathcal{S}(\mathcal{B}), \mathrm{mk}_{L_{\mathcal{B}}})$ onto $(\mathcal{S}(\mathcal{A}), \mathrm{mk}_{L_{\mathcal{A}}})$.

Tunnels between Quantum Compact Metric Spaces

We introduce Latrémolière's notion of tunnels.

Definition 3: Let $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ be two quantum compact metric spaces. A tunnel $\mathcal{T} = (\mathcal{D}, L_{\mathcal{D}}, \pi_1, \pi_2)$ between $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ is given by a quantum compact metric space $(\mathcal{D}, L_{\mathcal{D}})$ and two quantum isometries $\pi_1 : (\mathcal{D}, L_{\mathcal{D}}) \to (\mathcal{A}, L_{\mathcal{A}})$ and $\pi_2 : (\mathcal{D}, L_{\mathcal{D}}) \to (\mathcal{B}, L_{\mathcal{B}})$. The *extent* $\chi(\mathcal{T})$ of the tunnel $\mathcal{T}$ is the maximum of Hausdorff distances of metric spaces (where $\mathcal{S}(\mathcal{D})$ is given the Monge-Kantorovich metric):

$$\chi(\mathcal{T}) := \max\{d_H\big(\pi_1^*(\mathcal{S}(\mathcal{A})), \mathcal{S}(\mathcal{D})\big), d_H\big(\pi_2^*(\mathcal{S}(\mathcal{B})), \mathcal{S}(\mathcal{D})\big)\}.$$

Convergence in the GH Proprinquity

Definition 4: The (dual) Gromov-Hausdorff propinquity between two compact quantum metric spaces $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ is given by

$$\Lambda^*((\mathcal{A}, L_{\mathcal{A}}),(\mathcal{B}, L_{\mathcal{B}})) = \inf\{\chi(\mathcal{T}) : \ \mathcal{T} \text{ a tunnel from } (\mathcal{A}, L_{\mathcal{A}}) \text{ to } (\mathcal{B}, L_{\mathcal{B}})\}.$$

Result of Latrémolière (La) Let $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ be two compact quantum metric spaces. Then $\Lambda^*((\mathcal{A}, L_{\mathcal{A}}),(\mathcal{B}, L_{\mathcal{B}})) = 0$ if and only if $(\mathcal{A}, L_{\mathcal{A}})$ and $(\mathcal{B}, L_{\mathcal{B}})$ are fully quantum isometric. In addition, the (dual) Gromov-Hausdorff propinquity, $\Lambda^*(\cdot,\cdot)$, is a complete metric on the space of all full isometry classes of quantum compact metric spaces.

Spectral Triples: Introduction

Alain Connes first came up with the definition of spectral triples in the 1980's (C89). He noted that when X is a compact spin manifold, $\mathcal{A} = C(X)$, acting the Hilbert space of sections of the spinor bundle $H = L^2(X, S)$ by multiplication (S the spinor bundle over X) and $\not{D}$ is the Dirac operator defined on a dense subspace of H, then the triple $(\mathcal{A} = C(X),\ H,\ \not{D})$ together with the smooth subalgebra $C^\infty(X)$ of $C(X)$ can be used to recover the Riemannian metric on X.

A key idea behind spectral triples is to what extent is it possible to take the algebraic construct $(\mathcal{A},\ H,\ \not{D})$ with "smooth" subalgebra $\mathcal{A}^\infty \subseteq \mathcal{A}$ when $\mathcal{A}$ is a **noncommutative** unital C^*-algebra with a representation $\pi : \mathcal{A} \to B(H)$ and $\not{D}$ a "Dirac" operator, such that $[\not{D}, \pi(a)]$ can be extended to an element of $B(H)$, for all $a \in \mathcal{A}^\infty$.

Spectral Triples, as introduced by A. Connes:

Definition 5: A **spectral triple** $(\mathcal{A}, H, \not{D})$ consists of a unital C^*-algebra $\mathcal{A}$, a unital faithful representation π of $\mathcal{A}$ on a Hilbert space H, and a self-adjoint operator $\not{D} : \mathrm{dom}(\not{D}) \subseteq H \to H$, called the **Dirac operator** for the triple, such that

(ST1) the operator $\not{D}$ has compact resolvent, i.e. there exist $\lambda \in \mathbb{C} \setminus \sigma(\not{D})$ such that $R_\lambda(\not{D}) = (\not{D} - \lambda \mathrm{Id}_H)^{-1}$ is compact;

(ST2) there exists a dense $*$-subalgebra $\mathcal{A}^\infty$ of $\mathcal{A}$ such that for every $a \in \mathcal{A}^\infty$, $[\not{D}, \pi(a)]$ is densely defined and extends to a bounded operator on H. The subalgebra $\mathcal{A}^\infty$ is sometimes referred to as a **smooth subalgebra** of $\mathcal{A}$ with respect to $.\not{D}$.

If in addition there is an operator $J \in B(H)$ with $J = J^*$, $J^2 = \mathrm{Id}$, $\not{D} \circ J + J \circ \not{D} = 0$, and $J \circ \pi(a) = \pi(a) \circ J$, $\forall a \in \mathcal{A}$, we say the spectral triple is **even**. Otherwise, it is **odd**.

The Spectral Proprinquity for Spectral Triples

"Definition" 6: Let $(\mathcal{A}, H, \not{D})$ be an even or odd spectral triple, where $\mathcal{A}$ is a unital C^*-algebra. Given $a \in \mathrm{sa}(\mathcal{A})$, define $L_{\not{D}}$ by $L_{\not{D}}(a) = \|[\not{D}, \pi(a)]\|$. If $(\mathcal{A}, L_{\not{D}})$ is a compact quantum metric space, we say that $(\mathcal{A}, H, \not{D})$ is a *metric spectral triple*.

Several years ago, Latrémolière introduced a new metric on the on isomorphism classes of metric spectral triples, called the *spectral propinquity* Λ^{spec}. The definition involves defining a *modular propinquity* on certain metrical C^*-correspondences associated to the spectral triples. In the second step of the definition, a covariance condition with respect to the one-parameter group of unitary operators generated by $\not{D}$ is considered. The key consequence of this concept is that if $(\mathcal{A}, H, \not{D})$ and $(\mathcal{A}', H', \not{D}')$ are two metric spectral triples satisfying $\Lambda^{\mathrm{spec}}((\mathcal{A}, H, \not{D}), (\mathcal{A}', H', \not{D}')) = 0$, then there exists a full quantum isometry $\Theta : \mathcal{A} \to \mathcal{A}'$ implemented by a unitary $U : H \to H'$ so that $\pi'(\Theta(a)) = U\pi(a)U^*, \forall a \in \mathcal{A}$, and $U\not{D}U^* = \not{D}'$.

Inductive Limits of Spectral triples:

We give the definition of inductive limits of spectral triples in the sense of Floricel and Ghorbanpour (FG).

Definition 7 (FG): Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be an inductive limit of unital C^*-algebras with unital direct limit $\mathcal{A}_\infty = \overline{\cup_{n\in\mathbb{N}}\mathcal{A}_n}$. Suppose that $\{(\mathcal{A}_n, H_n, \not{D}_n)\}_{n\in\mathbb{N}}$ and $(\mathcal{A}_\infty, H_\infty, \not{D}_\infty)$ are spectral triples (either all odd or all even). We say that $(\mathcal{A}_\infty, H_\infty, \not{D}_\infty)$ is the inductive limit of $\{(\mathcal{A}_n, H_n, \not{D}_n)\}_{n\in\mathbb{N}}$ in the sense of Floricel and Ghorbanpour if:

1. $H_\infty = \overline{\cup_{n\in\mathbb{N}}H_n}$; (increasing union)
2. $\forall n \in \mathbb{N}$, the restriction of $\not{D}_\infty$ to dom $\not{D}_n$ is $\not{D}_n$;
3. $\forall n \in \mathbb{N}$, $\pi_n(\mathcal{A}_n)H_n \subseteq H_n$; i.e. for each n, H_n is reducing for $\mathcal{A}_n$.
4. If the spectral triples are even, the grading operator on H_∞ restricts to the grading operators on each H_n.

Spectral Triples Converging in the Spectral Proprinquity

It need not be the case that sequences of spectral triples be inductive limits in order to obtain a limiting spectral triple in the spectral propinquity, see for example the work of Landry, Lapidus and Latrémolière (LLL) for examples of spectral triples based on piecewise fractal curves converging to spectral triples based on the Sierpinski gasket in the spectral propinquity. There are also the examples due to Rieffel of spectral triples based on "fuzzy tori" (certain matrix algebras) converging to spectral triples on the sphere in the spectral propinquity, and these cannot be inductive limits, either.

But: in (FLP) we are able to come up with sufficient criteria that are fairly easy to check under which, given a metric spectral triple $(\mathcal{A}_\infty, H_\infty, \not{D}_\infty)$ that is the inductive limit of a sequence of metric spectral triples $\{(\mathcal{A}_n, H_n, \not{D}_n)\}_{n\in\mathbb{N}}$ as in Definition 7, we have

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((\mathcal{A}_n, H_n, \not{D}_n), (\mathcal{A}_\infty, H_\infty, \not{D}_\infty)) = 0.$$

Twisted discrete group algebras

Let Γ be a countable discrete group. Let σ be a multiplier on Γ (that is, a 2-cocycle on Γ taking on values in $\mathbb{T}$).
For any $f_1, f_2 \in \ell^1(\Gamma)$, the twisted convolution $*_\sigma$ is given by

$$f_1 *_\sigma f_2 : \gamma \in \Gamma \mapsto \sum_{\gamma_1 \in \Gamma} f_1(\gamma_1) f_2((\gamma_1)^{-1}\gamma)\, \sigma(\gamma_1, (\gamma_1)^{-1}\gamma),$$

and the adjoint operation by $f_1^* : \gamma \in \Gamma \mapsto \overline{f_1(\gamma^{-1})\, \sigma(\gamma, \gamma^{-1})}.$
In this way, we obtain the Banach $*$-algebra, denoted by $\ell^1(\Gamma, \sigma)$.
The left σ-regular representation of λ_σ of Γ on $\ell^2(\Gamma)$ is given by:

$$\lambda_\sigma(f)(g)(\gamma) \;=\; \sum_{\gamma_1 \in \Gamma} \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1) g(\gamma_1^{-1}\gamma),\ \gamma_1,\ \gamma\ \in\ \Gamma,$$

$\forall f \in \ell^1(\Gamma, \sigma),\ \forall g \in \ell^2(\Gamma).$

Twisted group C^*-algebras, full and reduced

Definition 7: If Γ is a discrete group and σ is a multiplier on Γ, the C^*-enveloping algebra of $\ell^1(\Gamma, \sigma)$ is denoted by $C^*(\Gamma, \sigma)$. If $\sigma \equiv 1$, then the associated C^*-algebra is denoted by $C^*(\Gamma)$.

The C^*-completion of $\ell^1(\Gamma, \sigma)$ in the norm given by the representation λ_σ is called **the reduced twisted group C^*-algebra associated to** σ and denoted by $C_r^*(\Gamma, \sigma)$. Note $C_r^*(\Gamma) := C_r^*(\Gamma, 1)$. If Γ is amenable, we have $C^*(\Gamma, \sigma) \cong C_r^*(\Gamma, \sigma)$ (ZM).

Noncommutative tori: twisted $\mathbb{Z}^2$ algebras

Definition 8: For $\theta \in [0,1)$, let σ_θ be the multiplier on $\mathbb{Z}^2$ defined by

$$\sigma_\theta\big((z_1, z_2),\ (y_1, y_2)\big) \ = \ e^{2\pi i\theta(z_1 y_2)},\ (z_1, z_2),\ (y_1, y_2) \ \in \ \mathbb{Z}^2.$$

Every multiplier on $\mathbb{Z}^2$ is cohomologous to one of the above type. Define $C^*(\mathbb{Z}^2, \sigma_\theta)$, also denoted by A_θ, to be the *noncommutative torus.*

It is standard knowledge dating to the 1960's-1970's that A_θ is the universal C^*-algebra generated by unitaries U_θ and V_θ such that

$$U_\theta V_\theta = e^{2\pi i\theta} V_\theta U_\theta. \tag{1}$$

Let Γ be a countable discrete group with a length function $\mathbb{L}$, which is a non-negative function satisfying "distance-like" conditions. Under appropriate conditions on the pair $(\Gamma, \mathbb{L})$, A. Connes in (C89) constructed a spectral triple on $C_r^*(\Gamma)$, with Dirac operator $\not{D}_{\mathbb{L}}$ defined on finitely supported functions on Γ by $\forall f \in C_C(\Gamma)$,

$$\not{D}_{\mathbb{L}}(f)(\gamma) \;=\; \mathbb{L}(\gamma) \cdot f(\gamma),\; \gamma \in \Gamma. \tag{2}$$

M. Rieffel (Rie), N. Ozawa and Rieffel (OR), and M. Christ and Rieffel (CR), in papers appearing in 2002, 2006 and 2017, gave a slightly different approach to the original construction of Connes. In particular, they allowed for the consideration of twisted reduced group C^*-algebras, and were interested in using a Dirac operator provided by length functions to give the unital C^*-algebras involved the structure of "compact quantum metric spaces", by means of seminorm on a dense subalgebra provided by taking the norm of the commutator with $\not{D}_{\mathbb{L}}$.

Spectral Triples for twisted and untwisted group C^*-algebras (ii)

Rieffel, Christ et al were able to build their spectral triples for a wide class of twisted and untwisted group C^*-algebras, including those coming from finitely generated nilpotent discrete groups, and noncommutative tori.

In the past few years, B. Long and W. Wu (LW) have extended these ideas, studying spectral triples on twisted group C^*-algebras of the form $(C_r^*(\Gamma, \sigma), \ell^2(\Gamma), \not{D}_{\mathbb{L}})$ coming from proper length functions $\mathbb{L}$ on discrete groups with the "bounded doubling property". However, they did not deal with the non-finitely generated case.

Twisted group C^*-algebras as compact quantum metric spaces

Let Γ be a discrete abelian group and σ be any multiplier on Γ. Suppose we are given length function $\mathbb{L}$ on Γ that has the "bounded doubling" property. We then obtain an odd spectral triple with Dirac operator $\not{D}_{\mathbb{L}}$, and an associated seminorm $L_{\not{D}}(a) = \|[\not{D}_{\mathbb{L}}, \pi_{\lambda_\sigma}(a)]\|$, which is a Leibniz Lip-norm on $C^*(\Gamma, \sigma)$, by work of Long and Wu.

The bounded doubling required to show that $L_{\not{D}_{\mathbb{L}}}$ gives a Monge-Kontorovich metric that induces the weak-$*$ topology on $\mathcal{S}(C^*(\Gamma, \sigma))$ is the most technical part of the proof, and involves showing that $\{f \in \mathrm{dom}(L_{\not{D}_{\mathbb{L}}}) : L_{\not{D}_{\mathbb{L}}}(f) \leq 1 \text{ and } \tau(f) = 0\}$ is totally bounded in $C^*(\Gamma, \sigma)$.

Noncommutative solenoids: background and structure

Let p be a fixed prime number. Let $\mathbb{Z}[\frac{1}{p}] = \cup_{j=0}^{\infty}(\frac{1}{p^j}(\mathbb{Z})) \subset [\mathbb{Q}]$. Note $\mathbb{Z}[\frac{1}{p}]$ is the direct limit of the groups $\frac{1}{p^j}(\mathbb{Z})$. The group $\mathbb{Z}[\frac{1}{p}]$ is **not** finitely generated.

The Pontryagin dual of $\mathbb{Z}[\frac{1}{p}]$ is the compact inverse limit abelian group $\{(z_i)_{i=0}^{\infty} : \ z_i \in \mathbb{T}, \ (z_{i+1})^p = z_i, \ \forall i\}$. This group is called the p-solenoid, and denoted by $\mathcal{S}_p$.

We now construct spectral triples for twisted group C^*-algebras associated to the Cartesian product $\left(\mathbb{Z}[\frac{1}{p}]\right)^2 := \Gamma$. We can write Γ as the direct limit of $\Gamma_j \ = \ \frac{1}{p^j}[(\mathbb{Z})^2]$.

Noncommutative solenoids, cont.

Modeling the terminology on the noncommutative tori $C^*(\mathbb{Z}^2, \sigma)$, we define *noncommutative solenoids*.

Definition 9: (LP) Fix a prime p, and let σ be any multiplier on $\Gamma = \left(\mathbb{Z}[\frac{1}{p}]\right)^2$. Then the twisted group C^*-algebra $C^*(\Gamma, \sigma)$ is called a noncommutative solenoid of dimension 2. It is possible to construct a multiplier σ_θ cohomologous to σ from a sequence $(\theta_i)_{i\in\mathbb{N}} \in \Pi_{i=1}^{\infty}[0,1)_i : \ \forall i \in \mathbb{N}, p\theta_{i+1} = \theta_i \mod p$. Denote the noncommutative solenoid $C^*(\left(\mathbb{Z}[\frac{1}{p}]\right)^2, \sigma_\theta)$ by $\mathcal{A}_\theta^{\mathcal{S}}$.

$\mathcal{A}_\theta^{\mathcal{S}}$ will be $*$-isomorphic to the direct limit C^*-algebra below:

$$\varinjlim_{n\in\mathbb{N}} A_{\theta_{2n}} : \quad A_{\theta_0} \xrightarrow{\varphi_0} A_{\theta_2} \xrightarrow{\varphi_1} A_{\theta_4} \xrightarrow{\varphi_2} \cdots A_{\theta_{2n}} \xrightarrow{\varphi_n} A_{\theta_{2n+2}} \cdots$$

Here each $A_{\theta_{2n}}$ is the rotation algebra generated by unitaries U_{2n} and V_{2n} and $\varphi_n(U_{2n}) = U_{2n+2}^p$; $\varphi_n(V_{2n}) = V_{2n+2}^p$.

A proper length function on $\mathbb{Z}[\frac{1}{p}]^2$ with bounded doubling

Lemma 1: (FLLP), (FLP) Fix a prime p. Define the "scale" function $\mathbb{F}_p : \mathbb{Z}\left[\frac{1}{p}\right]^2 \to [0, \infty)$ by

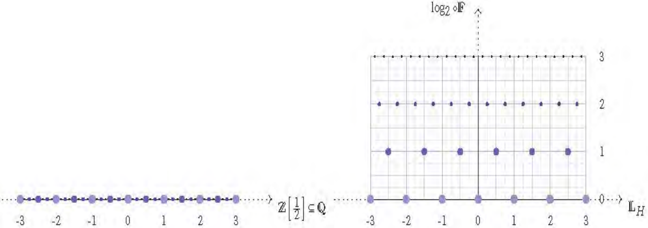
$$\mathbb{F}_p((r_1, r_2)) \;=\; p^{\text{scale}(r_1, r_2)},$$

where $\text{scale} : \mathbb{Z}\left[\frac{1}{p}\right]^2 \to \mathbb{N}$ is defined by $\text{scale}(r_1, r_2) \;=\; \min\{n \in \mathbb{N} : \; (r_1, r_2) \;\in\; [\frac{1}{p^n}\mathbb{Z}]^2\}$. Define $\mathbb{L}_p : \mathbb{Z}\left[\frac{1}{p}\right]^2 \to [0, \infty)$ by

$$\mathbb{L}_p(r_1, r_2) \;=\; \max\{\|(r_1, r_2)\|_2,\; \mathbb{F}_p(r_1, r_2)\},$$

where $\|\cdot\|_2$ is the restriction of the standard Euclidean norm on $\mathbb{R}^2$. Then $\mathbb{L}_p$ is a length function on $\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^2$ that is proper, unbounded, and has bounded doubling.

The contributions of ordinary distance and “scale" to our length on $\mathbb{Z}[\frac{1}{p}]$



(A) The geometry of $\mathbb{Z}\left[\frac{1}{2}\right]$ for $|\cdot|$ in $\mathbb{Q}$

(B) The geometry of $\mathbb{Z}\left[\frac{1}{2}\right]$ using $\mathbb{F}$

Odd Spectral triples for nc solenoids

Corollary 1: (FLLP), (FLP) Fix a prime p. Define the length function with bounded doubling $\mathbb{L}_p : \left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^2 \to [0,\infty)$ as in the previous Lemma, and define the Dirac operator $\not{D}_p$ on a dense subspace of $\ell^2(\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^2)$ by $\not{D}_p = D_{\mathbb{L}_p}$. Then taking $\Gamma = \left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^2$, $(C^*(\Gamma,\sigma_\theta),\ell^2(\Gamma),\not{D}_p)$ is an odd metric spectral triple, $\forall \theta \in \Omega_p$.

Proof: This follows the construction of the length function with bounded doubling, from results of Christ and Rieffel (CR) and Long and Wu (LW).

Notation for Even Spectral Triples and Main Theorem:

Let $\{\Gamma_n\}$ be an increasing sequence of discrete abelian groups with $\Gamma = \bigcup_{n\in N} \Gamma_n$, and $\mathbb{L}_H$ a length on Γ, $\mathbb{F}$ a *scale* on Γ. For our next result on convergence of spectral triples in the spectral propinquity, we use the following notation:

(*i*) $H_\infty = \ell^2(\Gamma) \otimes \mathbb{C}^2$, $\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$\gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $J = \mathrm{id}_{\ell^2(\Gamma)} \otimes i\gamma_1\gamma_2$, and

$\not{D} = M_{\mathbb{L}_H} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2$ on

$\left\{\xi \in \ell^2(\Gamma) \otimes \mathbb{C}^2 : \sum_{g\in\Gamma}(\mathbb{L}_H(g)^2 + \mathbb{F}(g)^2)\|\xi(g)\|_E^2 < \infty\right\}$.

(*ii*) $H_n = \ell^2(\Gamma_n) \otimes \mathbb{C}^2$ is identified with the subspace of Γ_n-supported vectors in $\ell^2(\Gamma) \otimes \mathbb{C}^2$,

(*iii*) $\not{D}_n$ is the restriction of $\not{D}$ to $\mathrm{dom}\,\not{D} \cap \left(\ell^2(\Gamma_n) \otimes \mathbb{C}^2\right)$,

(*iv*) $C^*(\Gamma, \sigma)$ and $C^*(\Gamma_n, \sigma)$ act via their left regular σ-projective representations on $\ell^2(\Gamma)$ and $\ell^2(\Gamma_n)$, respectively.

Convergence of the Compact Quantum Metric Space in GH Proprinquity

Theorem 1:(FLP): Let $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ be an abelian discrete group, arising as the union of a strictly increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subgroups of Γ. Let σ be a 2-cocycle of Γ and $\mathbb{L}_H$ a length function on Γ such that $\lim_{n\to\infty} \mathrm{Haus}[\mathbb{L}_H](\Gamma_n, \Gamma) = 0$ (pointed GH distance for metric spaces), and whose restriction to Γ_n is proper for all $n \in \mathbb{N}$. Assume $\mathrm{scale} : \mathbb{N} \to [0, \infty)$ is a strictly increasing, unbounded function such that, if we set

$$\mathbb{F} : g \in \Gamma \longmapsto \mathrm{scale}(\min\{n \in \mathbb{N} : g \in \Gamma_n\}),$$

then the proper length function $\mathbb{L} := \max\{\mathbb{L}_H, \mathbb{F}\}$ has the *bounded doubling property*. Then the even spectral triple $(C^*(\Gamma, \sigma), \ell^2(\Gamma) \otimes \mathbb{C}^2, \not{D})$ is metric, and is the inductive limit of the spectral triples $\{(C^*(\Gamma_n, \sigma), \ell^2(\Gamma_n) \otimes \mathbb{C}^2, \not{D}_n) : n \in \mathbb{N}\}$. Moreover, $\lim_{n\to\infty} \Lambda^*(((C^*(\Gamma_n, \sigma), L_{\not{D}_n}), (C^*(\Gamma, \sigma), L_{\not{D}})) = 0.$

Key Technical Point in Proof of Theorem 1

A key step in the proof of Theorem 1 is to show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, then:

$$\forall a \in \mathrm{dom} L_{\not{D}}, \exists b \in \mathrm{dom} L_{\not{D}_n} : L_{\not{D}_n}(b) \leq L_{\not{D}}(a) \ \& \|a - b\| \leq \varepsilon L_{\not{D}}(a),$$

and

$$\forall b \in \mathrm{dom} L_{\not{D}_n}, \exists a \in \mathrm{dom} L_{\not{D}} : L_{\not{D}}(a) \leq L_{\not{D}_n}(b) \ \& \ \|a - b\| \leq \varepsilon L_{\not{D}_n}(b).$$

This is for $\mathrm{dom} L_{\not{D}_n} \subset C^*(\Gamma_n, \sigma)$, and $\mathrm{dom} L_{\not{D}} \subset C^*(\Gamma, \sigma)$, where Γ has a length function with the right properties. The needed estimates then allow comparison of seminorms coming from the Dirac operators in the odd triples coming just from the length function, and the seminorms coming from the Dirac operators on the even triples, for Γ_n, and Γ.

Situation where Main Theorems can be Applied

We can take the noncommutative solenoid $C^*((\mathbb{Z}[\frac{1}{p}])^2, \sigma_\theta)$, write it as $\varinjlim_{n\in\mathbb{N}} A_{\theta_{2n}}$, and then recall

$$A_{\theta_{2n}} \cong C^*(\left(\tfrac{1}{p^n}\mathbb{Z}\right)^2, \mathrm{res}(\sigma_\theta)), \forall n \in \mathbb{N}.$$

Taking $H_n = \ell^2(\left(\frac{1}{p^n}\mathbb{Z}\right)^2) \otimes \mathbb{C}^2$, and restricting the Dirac operator $\not{D}$ to the appropriate dense subspace of H_n to get the Dirac operator $\not{D}_n$, we can represent each $A_{\theta_{2n}}$ on H_n. This gives for every $n \in \mathbb{N}$ a finitely summable metric spectral triple $(A_{\theta_{2n}}, H_n, \not{D}_n)$ with smooth subalgebra $C_C(\mathbb{Z}^2, \sigma_{\theta_{2n}})$, to which the above theorems can be applied. Not only is the NC solenoid the direct limit of the $\{A_{\theta_{2n}}\}$, but it is also the limit in the (dual) GH-propinquity and one can construct spectral triples for each $A_{\theta_{2n}}$, and obtain convergence in the spectral propinquity as well.

The Even Spectral Triples as Limits in the Spectral Propinquity

By doing the appropriate book-keeping and checking the requirements of Theorem 1, we obtain:

Corollary 2 (FLP): Fix a prime p. Let $\Gamma = \left(\mathbb{Z}\left[\frac{1}{p}\right]\right)^2$ and for each $n \in \mathbb{N}$, set $\Gamma_n = \frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$. For every $\theta \in \Omega_p$, the even spectral triple triple $(\mathcal{A}_\theta^{\mathcal{S}}, \ell^2(\Gamma) \otimes E, \not{D})$ with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$ can be constructed as the limit in the sense of the spectral proprinquity of Latrémolière of the spectral triples $\{(A_{\theta_{2n}}, H_n, \not{D}_n)\}_{n\in\mathbb{N}}$ with corresponding dense subalgebras $\{C_C(\mathbb{Z}^2, \sigma_{\theta_{2n}})\}_{n\in\mathbb{N}}$.

A Spectral Triple for Bunce-Deddens algebras of "bounded" type

We review the Bunce-Deddens algebras (BD) and concentrate on constructing spectral triples on those of "bounded" type. We note that metric spectral triples on Bunce-Deddens algebras have also been considered by Aguilar, Latrémolière, and Rainone (ALR) .

Definition 10: (FLP) Let $\mathcal{P}$ be the set of all sequences $(N_k)_{k\in\mathbb{N}}$ of positive integers such that $\frac{N_{k+1}}{N_k}$ is a prime number for all $k \in \mathbb{N}$. We say a sequence $(N_k) \in \mathcal{P}$ is *bounded* if $(\frac{N_{k+1}}{N_k})$ contains only finitely many primes $\{p_1 < p_2 < \cdots < p_M\}$, for some $M \in \mathbb{N}$. Denote the set of all bounded sequences in $\mathcal{P}$ by $\mathcal{B}P$. Given $(N_k)_{k\in\mathbb{N}} \in \mathcal{B}P$, define an embedding

$$\rho_k : \mathbb{Z}_{N_k} \to \mathbb{Z}_{N_{k+1}}$$

by $\rho_k([j]_{\text{mod } N_k}) = [(\frac{N_{k+1}}{N_k}) \cdot j]_{\text{mod } N_{k+1}}$ and let

$$\mathbb{Z}[(N_k)] = \lim_{k\to\infty} \rho_k(\mathbb{Z}_{N_k}).$$

Bunce-Deddens algebras continued

One can show that $\mathbb{Z}[(N_k)]$ is isomorphic to the discrete group

$$\mathcal{Z}[(N_k)] \;=\; \{z \in \mathbb{T} :\; \exists k \in \mathbb{N} \text{ s.t. } z^{N_k} = 1\}.$$

We now fix $(N_k)_{k\in\mathbb{N}} \in \;\mathcal{B}P$, and let $\Gamma \;=\; \mathbb{Z}[(N_k)] \times \mathbb{Z}$. It can be shown that every multiplier on Γ is cohomologous to one of the form $\sigma_\nu : \Gamma \to \mathbb{T}$ given by

$$\sigma_\nu((v_1, m_1), (v_2, m_2)) \;=\; [\nu(v_2)]^{m_1} \;=\; v_2(\nu^{m_1}),$$

where $\nu \in \cong \;\widehat{\mathbb{Z}[(N_k)]}$, a compact totally disconnected group of Cantor type.
Then $C^*(\Gamma, \sigma_\nu)$ can be expressed as a direct limit

$$\lim_{k\to\infty} C^*(\mathbb{Z}_{N_k} \times \mathbb{Z}, \sigma_{\nu_k}).$$

These latter algebras are $*$-isomorphic to direct limits of $M_{N_k}(C(\mathbb{T}))$, and hence are C^*-algebras of Bunce-Deddens type that we call “bounded".

Bunce-Deddens algebras, part (iii)

Indeed viewing $\nu \in \widehat{\mathbb{Z}[(N_k)]}$ as the "generator" for an "odometer shift" on $\widehat{\mathbb{Z}[(N_k)]}$, by work of Exel and P., $C^*(\mathbb{Z}[(N_k)] \times \mathbb{Z}, \sigma_\nu)$ is the Bunce-Deddens algebra corresponding to the "supernatural numeral"

$$\mathrm{n} := \left(p^{\#\{k \in \mathbb{N},\ \frac{N_{k+1}}{N_k} = p\}} \right).$$

Corollary 3 (FLP): Define a "scale" function $\mathbb{F}_{\mathbb{T}}$ on $\mathbb{Z}[(N_k)]$ by:

$$\mathbb{F}_{\mathbb{T}}(e^{2\pi i \frac{a}{n}}) = 0, \text{ if } a = 0, \text{ and } \mathbb{F}_{\mathbb{T}}(e^{2\pi i \frac{a}{n}}) = n, \text{ if } \gcd(a, n) = 1.$$

Define $\mathbb{L}_H : \mathbb{Z}[(N_k)] \times \mathbb{Z} \to [0, \infty)$ by $\mathbb{L}_H(z, n) = |z| + |n|$. Then $\Gamma = \mathbb{Z}[(N_k)] \times \mathbb{Z}$, and defining $\not{D}_{BD} = M_{\mathbb{L}_H} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2$, $(C^*(\Gamma, \sigma_\nu), \ell^2(\Gamma) \otimes \mathbb{C}^2, \not{D}_{BD})$ is an even spectral triple on bounded Bunce-Deddens algebras.

References:

- K. Aguilar, F. Latrémolière, and T. Rainone, "Bunce-Deddens algebras as quantum Gromov-Hausdorff distance limits of circle algebras", Integral Equations Operator Theory **94** (2022), Paper no. 2; (ALR).
- M. Christ and M. Rieffel, "Nilpotent group C^*-algebras as compact quantum metric spaces", Canad. Math. Bull. 60 (2017), 77–94 (CR).
- A. Connes, "Compact metric spaces, Fredholm modules, and hyperfiniteness", Ergodic Theory Dynam. Systems 9 (1989), 207–220; (C89).

References, continued:

- A. Connes, "On the spectral characterization of manifolds", J. Noncommut. Geom. 7 (2013), 1–82; (C13).
- C. Farsi, T. Landry, N. Larsen, and J. Packer, "Spectral triples for moncommutative solenoids via length functions and inductive limits, and a noncommutative eiener's Lemma", submitted (2022), arXiv:2212.07470 (FLLP).
- C. Farsi, F. Latrémolière and J. Packer, "Convergence of inductive sequences of spectral triples for the spectral propinquity", submitted (2023), arXiv:2301.00274; (FLP).
- R. Floricel and A. Ghorbanpour, "On inductive limit spectral triples" Proc. Amer. Math. Soc. 147 (2019), 3611–3619; (FG).
- T. Landry, M. Lapidus, and F. Latrémolière, "Metric approximatios of the spectral triple on the Sierpinki gasket and other fractals,", Adv.Math. 385 (2021), Paper No. 107771; (LLL).

References, continued:

- F. Latrémolière, "The Quantum Gromov-Hausdorff Propinquity", Trans. Amer. Math. Soc. 368 (2016), 365–411; (La).
- F. Latrémolière, "Convergence of spectral triples on fuzzy tori to spectral triples on quantum tori", Comm.Math. Phys. 388 (2021), 1049–1128; (La1)
- F. Latrémolière, "The Gromov-Hausdorff propinquity for metric spectral triples", Adv. Math. 404 (2022), Paper No. 108393; (La2)
- F. Latrémolière and J. Packer, "Noncommutative solenoids", New York J. Math 24A (2018), 155–191.; (LP).
- F. Latrémolière and J. Packer,"Noncommutative solenoids and the Gromov-Hausdorff propinquity", Proc. Amer. Math. Soc. 145 (2017), 2043–2057; (LP2).

References, continued:

- B. Long and W. Wu, "Twisted bounded-dilation group C^*-algebras as C^*-metric algebras", Science China Mathematics 64 (2021), 547–572, (LW)
- N. Ozawa and M. Rieffel, "Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces", Canad. J. Math. Vol. 57, 2005, 1056–1079; [OR].
- M. Rieffel, " Group C^*-algebras as compact quantum metric spaces", Doc. Math. 7 (2002), 605–651; (Rie).
- G. Zeller-Meier, "Produits croisés d'une C^*-algèbre par un groupe d' Automorphismes, J. Math. Pures et Appl. 47 (1968), 101–239; (ZM).