Supercritical equilibrium states of the right $ax + b$ system and their factorizations

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Based on joint work with Marcelo Laca

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Our objective is to investigate the statistical behaviour of arithmetic using techniques from quantum statistical mechanics.

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- symmetries of equilibrium states;
- partition functions, $Z(\beta)$.

[Laca-Raeburn] The *left $ax + b$ system* is the universal C*-algebra $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^{\times})$ generated by a unitary U (addition) and isometries V_a (multiplication), satisfying the relations:

1 $V_a V_b = V_{ab}$

2 $V_a V_b^* = V_b^* V_a$ when $\gcd(a, b) = 1$

3L $V_a U = U^a V_a$

4L $V_a^* U^k V_a = 0$ if $1 \leq k < a$,

spanned by elements of the form $U^n V_a V_b^* U^m$, $n, m \in \mathbb{Z}$ and $a, b \in \mathbb{N}^{\times}$, with the $\mathbb{R}$-action defined on spanning elements by

$$\sigma_t(U^n V_a V_b^* U^m) = \left(\frac{a}{b}\right)^{it} U^n V_a V_b^* U^m.$$

The $ax + b$ Systems

[An Huef-Laca-Raeburn] The *right $ax+b$ system* is the universal C*-algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ generated by a unitary U (addition) and isometries V_a (multiplication), satisfying the relations:

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Let A be a C*-algebra, σ_t a strongly-continuous $\mathbb{R}$-action, and $\beta \in \mathbb{R}$.

Definition

A KMS_β *state* on (A, σ_t) is a state ϕ such that

$$\phi(xy) = \phi(y\sigma_{i\beta}(x))$$

for all x, y in a dense subalgebra of A.

β is the inverse temperature. The set of KMS_β states is a Choquet simplex.

Different KMS structures

(Left) The simplex of KMS_β states is affinely isomorphic to:

- [LR] probability measures on the circle $\mathbb{T}$, when $\beta \in (2, \infty)$;
- [LR] a point, when $\beta \in [1, 2]$;
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(Right) The simplex of KMS_β states is affinely isomorphic to:

- [aHLR] probability measures on the circle $\mathbb{T}$, when $\beta \in (1, \infty)$.
- Open for $\beta \leq 1$.

We provide a partial answer for $\beta \leq 1$.

Theorem (Laca-S.)

For $\beta \in (0,1]$, there is a simplex of KMS_β states affinely isomorphic to the simplex of probability measures on the compact space $\mathbb{N}^\times \cup \{\infty\}$.

The extremal KMS_β states $\psi_{n,\beta}$ corresponding to δ_n, $n \in \mathbb{N}^\times$ are also extreme in the simplex of all KMS_β states, and the GNS representation is a Type III_1 factor.

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Tentatively, this simplex contains *every* KMS_β state if the *periodic zeta function*

$$\zeta(\beta, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\beta},$$

converges conditionally at $\beta = 1$ for all $z \in \mathbb{T} \setminus \{1\}$ (awaiting some details).

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- $\mathbb{T}$ is the circle, $\Delta_a = \{d \in \mathbb{N}^\times : d|a\}$, and $X_a = \mathbb{T} \times \Delta_a$.
- When $a|b$, $f_{a,b} : X_b \to X_a$, $f_{a,b}(z, d) = (z^{d/\gcd(a,d)}, \gcd(a, d))$. These satisfy $f_{a,b} \circ f_{b,c} = f_{a,c}$.

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The isometries $V_a U^n V_a^*$ are functions supported on $f_a^{-1}(\mathbb{T} \times \{a\})$ and the co/isometries V_a and V_a^* act by partial homeomorphisms of X.

ψ_n explained

This bears a striking resemblance to the Bost-Connes System $(C_{\mathbb{Q}}, \sigma_t)$, wherein KMS_β states factor through a conditional expectation to $C(\hat{\mathbb{Z}})$, $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/a\mathbb{Z}$.

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Proposition (LS)

*The extremal KMS_β state $\psi_{n,\beta}$ factors through a composition of $\mathbb{R}$-equivariant *-homomorphisms*

$$\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/n\mathbb{Z})) \to C_{\mathbb{Q}},$$

where $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ is the universal C-algebra gen. by U and V_a with the extra relation*

4R $U^n = 1$.

Theorem (LS)

For every $m \in \mathbb{N}^\times$*, the simplex of* KMS_β *states of* $(\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})), \sigma_t)$ *is affinely isomorphic to*

- *probability measures on* $\mathbb{Z}/m\mathbb{Z}$ *for* $\beta \in (1, \infty)$*;*
- *probability measures on* Δ_m *for* $\beta \in (0, 1]$*.*

The group of the units $(\mathbb{Z}/m\mathbb{Z})^*$ *acts on* $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ *by* $\mathbb{R}$*-equivariant isomorphisms. The induced action on the* KMS_β *states leaves states invariant when* $\beta \in (0, 1]$ *and acts transitively on extremal* KMS_β *with the same gcd*$(m, \cdot)$ *when* $\beta \in (1, \infty)$*.*

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Thank you!