The lattice of C*-covers of an operator algebra

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MacEwan University

May 23, 2023

Overview

Non-selfadjoint operator algebras

Reminder.

A C*-algebra is a subspace of $B(H)$ that is

- closed to multiplication,
- $*$-closed
- and norm-closed.

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("Abstract" operator algebras are characterized by matrix norms (Blecher-Ruan-Sinclair).)

Basic operator algebras

Example.

M_n is a C*-algebra.
$B(H)$ is, too.

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If X is a compact space, then $C(X)$ is a C*-algebra.

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If X is a nice compact domain in $\mathbb{C}$, the *holomorphic functions* $A(X^\circ)$ are an operator algebra.

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Every group defines a **group C*-algebra(s)**.

Every **semigroup** determines a **semigroup operator algebra**.

Overview

C*-covers and C*-envelopes

Fundamental Problem.

Every operator algebra A generates a C*-algebra $C^*(A)$.

C*-covers and C*-envelopes

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Example.

$C^*(T_n) = M_n$.

But (completely isometrically) isomorphic copies of an operator algebra can generate different C*-algebras!

Examples of C*-covers

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$C^*(T_n) = M_n$.

Example.

$$T_2 \xrightarrow{\varphi} \mathbb{C} \oplus M_2$$
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \left(a, \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right)$$

is an isomorphism onto a copy of T_2, with $C^*(\varphi(T_2)) = \mathbb{C} \oplus M_2 \ncong M_2$.

C*-covers and C*-envelopes

Definition.

A **C*-cover** of an operator algebra A is a pair (C, ι) where

$$\iota : A \hookrightarrow C = C^*(\iota(A))$$

is a (completely isometric) embedding.

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Definition-Theorem.
(Arveson, Hamana, Dritschel-McCullough, Arveson (again), Davidson-Kennedy)

The **C*-envelope** is the unique **smallest** C*-cover of A, usually denoted $C^*_{\text{min}}(A)$. It exists for any A.

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(There is also a "$C^*_{\max}(A)$".)

The Disk Algebra

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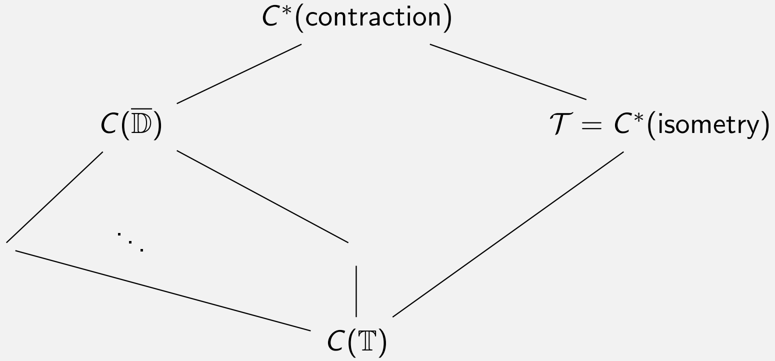
C*-covers of $A(\mathbb{D})$



C*-cover uniqueness

Question.

If two operator algebras have "the same C*-covers", are they isomorphic? If not, how different can they be?

C*-cover uniqueness

Question.

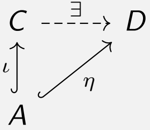
If two operator algebras have "the same C*-covers", are they isomorphic? If not, how different can they be?

(Two different operator algebras can have the same C*-envelope, or the same C*-max.)

Ordering of C*-covers

Definition.

For C*-covers, we say that $(C, \iota) \geq (D, \eta)$ if



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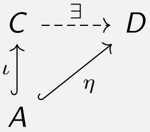

Theorem. (Hamidi, '19) (Thompson, '21)

The order $\leq$ makes the set of C*-covers of A a **complete lattice**.

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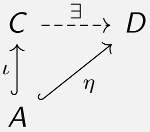

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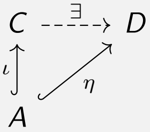

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(A partially ordered set where all subsets have "sups" and "infs".)

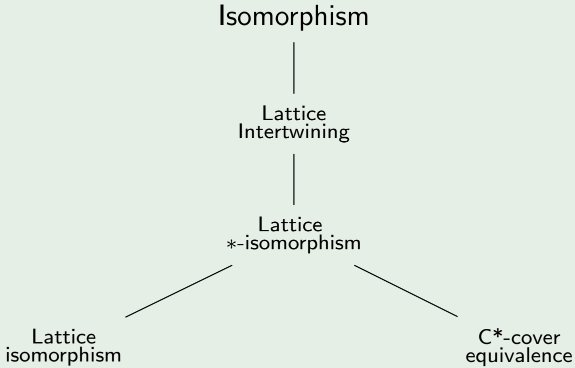
Call it C*-Lat(A).

Overview

C*-cover equivalences

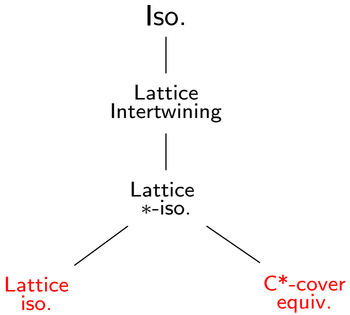
Theorem. (H-Ramsey '23)

The equivalence relations of "same C*-covers"



between two operator algebras are all distinct.

C*-cover equivalences for A and B



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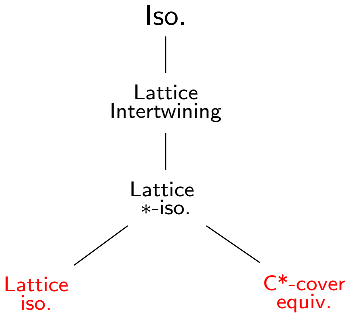
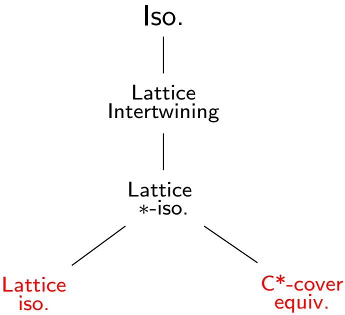


C*-cover equivalence

Every C*-cover of A is $*$-isomorphic to a C*-cover for B, and vice versa.

C*-cover equivalences for A and B



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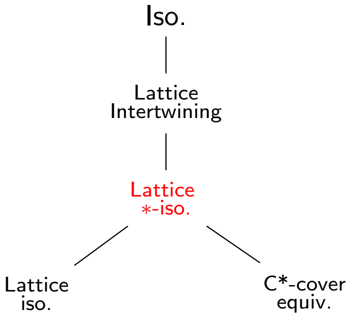
Every C*-cover of A is $*$-isomorphic to a C*-cover for B, and vice versa.

Lattice isomorphism

The lattices $\mathrm{C}^*\text{-Lat}(A)$ and $\mathrm{C}^*\text{-Lat}(B)$ are order isomorphic.

C*-cover equivalences for A and B



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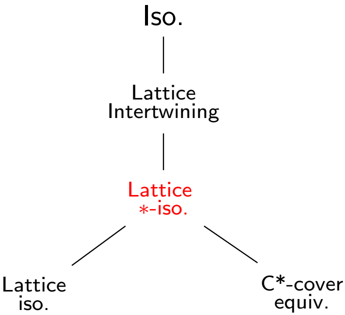
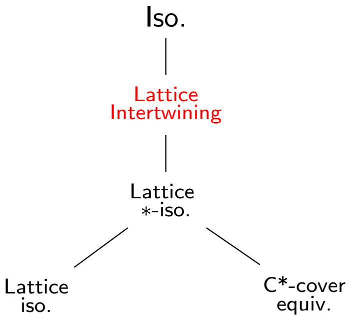


Lattice ∗-isomorphism

C*-Lat(A) and C*-Lat(B) are order isomorphic via an isomorphism that associates isomorphic C*-algebras.

C*-cover equivalences for A and B



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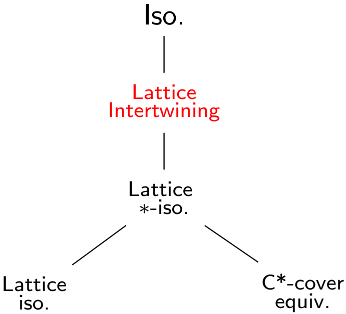
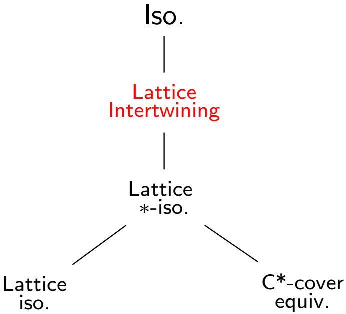


Lattice intertwining.

$$\begin{array}{ccc} (C_1, \iota_1) & \overset{\pi_{C_1}}{\dashrightarrow} & (D_1, \eta_1) \\ \downarrow & & \downarrow \\ (C_2, \iota_2) & \overset{\pi_{C_2}}{\dashrightarrow} & (D_2, \eta_2) \end{array}$$

(in C*-Lat(A)) (in C*-Lat(B))

C*-cover equivalences for A and B



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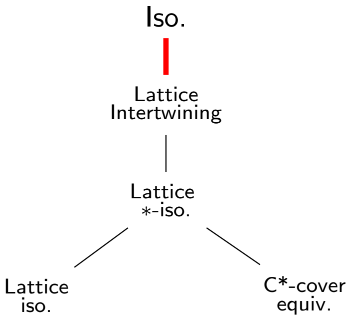
(in C*-Lat(A)) (in C*-Lat(B))

(Natural isomorphism.)

Separating Examples



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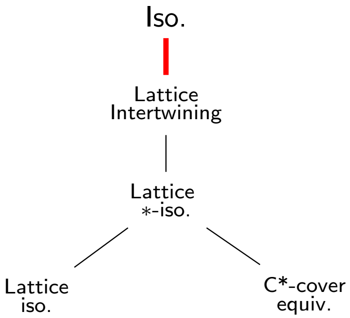

Fact.

If (C, ι) is a C*-cover for A, then (C, ι^*) is a C*-cover for A^*,

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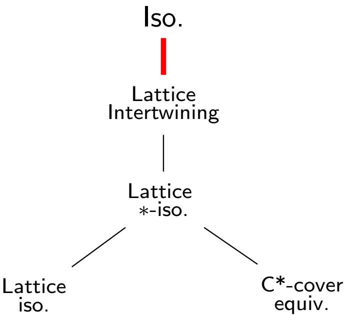

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If (C, ι) is a C*-cover for A, then (C, ι^*) is a C*-cover for A^*,

so A and A^* are intertwined.

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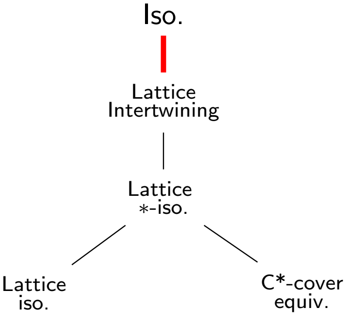

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But maybe $A \ncong A^*$!

$$\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \ncong \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{pmatrix}.$$

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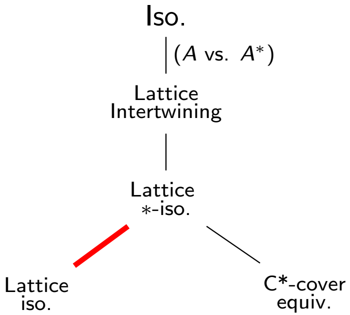

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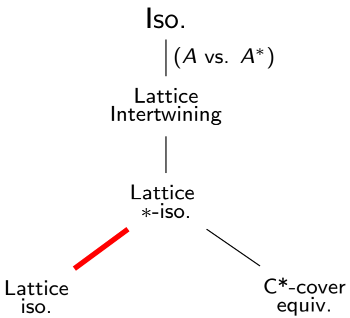

Fact.

If C is a C*-algebra, then

$$\text{C}^*\text{-Lat}(C) = \{(C, \text{id})\}.$$

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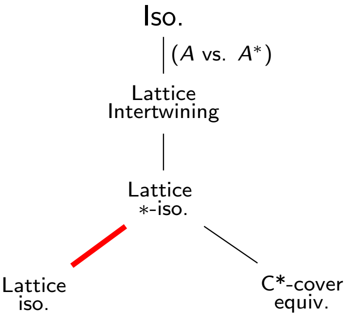

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So all C*-algebras are lattice iso.

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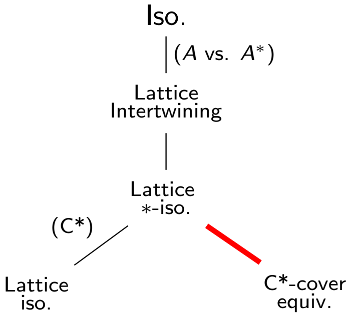

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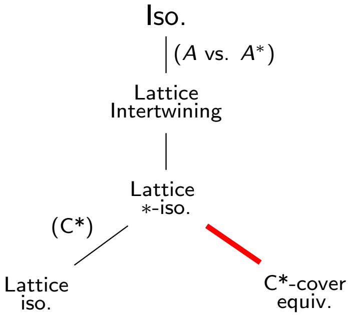

Fact.

$$\text{C}^*\text{-Lat}(A \oplus B) = \text{C}^*\text{-Lat}(A) \times \text{C}^*\text{-Lat}(B).$$

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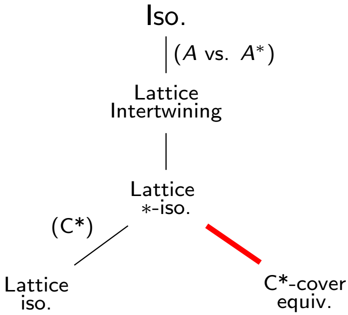

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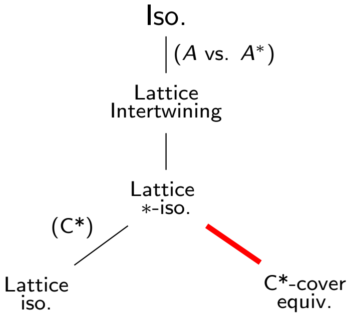

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So A and $A \oplus$ (C*-alg.) are lattice iso.

$$A \bigoplus (\text{all C*-covers of } A)$$

and

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are C*-cover equiv. but not lattice iso.

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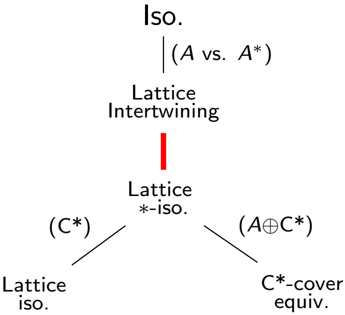

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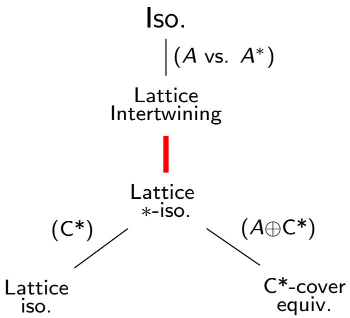

Theorems. (H-Ramsey '23)

What does

$$\mathrm{C}^*\text{-Lat}(A \otimes_\alpha B)$$

look like?

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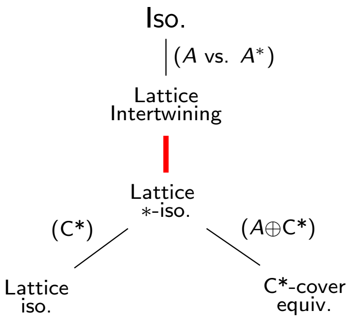

Theorems. (H-Ramsey '23)

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look like?

If B is a nuclear simple C*-algebra, then

$$\mathrm{C}^*\text{-Lat}(A \otimes B) \text{ `` } = \text{ '' } \mathrm{C}^*\text{-Lat}(A) \otimes B.$$

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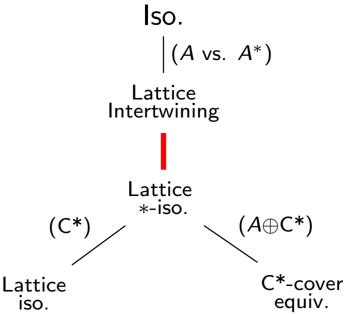

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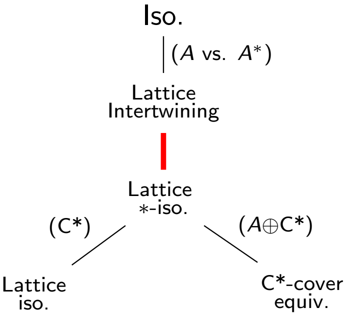

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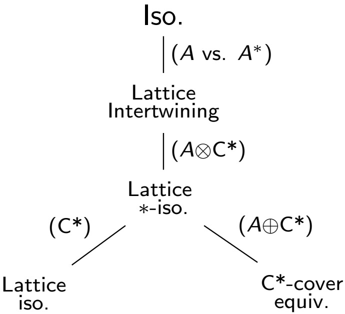

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Only one C*-cover?

Open Question.

If A has a one-point C*-lattice ($C^*_{\max}(A) = C^*_{\min}(A)$), is A a C*-algebra?

Only one C*-cover?

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If A has a one-point C*-lattice ($C^*_{\text{max}}(A) = C^*_{\text{min}}(A)$), is A a C*-algebra?

(Can A be lattice iso. to a C*-algebra?)

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(Kirchberg-Wassermann '99). The analogous question for operator *systems* is false.

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(Kirchberg-Wassermann '99). The analogous question for operator *systems* is false.

Partial Results (H-Ramsey '23)

If A is Dirichlet, hyperrigid, or embeds in finite dimensions, then A cannot have a one-point lattice unless it is a C*-algebra.

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Open Question.

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Related Questions.

Can C*-Lat(A) be

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Related Questions.

Can C*-Lat(A) be

- finite?

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Related Questions.

Can C*-Lat(A) be

- finite?
- of any cardinality?

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Can C*-Lat(A) be

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- any complete lattice you want?

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Related Questions.

Can $\mathrm{C}^*\text{-Lat}(A)$ be

- finite?
- of any cardinality?
- any complete lattice you want?

Another result. (H-Ramsey '23)

Construction of a simple operator algebra which is not similar to a C*-algebra.

Overview

Future Directions.

Undergraduate Project (J. Rumball, MacEwan, Riipen Level UP)

Theoretical or computer classification of finite-dimensional operator algebras which satisfy the "**semi-Dirichlet property**".

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This ties into

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Theoretical or computer classification of finite-dimensional operator algebras which satisfy the "**semi-Dirichlet property**".

This ties into

Theorem. (H-Ramsey, Forthcoming)

The semi-Dirichlet C*-covers of an operator algebra form a complete lattice, and there is a maximal semi-Dirichlet C*-cover.

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Undergraduate Project (J. Rumball, MacEwan, Riipen Level UP)

Theoretical or computer classification of finite-dimensional operator algebras which satisfy the "**semi-Dirichlet property**".

This ties into

Theorem. (H-Ramsey, Forthcoming)

The semi-Dirichlet C*-covers of an operator algebra form a complete lattice, and there is a maximal semi-Dirichlet C*-cover.

More forthcoming work. (H-Ramsey)

The **RFD** C*-covers do not form a complete lattice.
Are they even a lattice?

Future Directions.

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Thank you!