Majorization in C^*Algebras

Paul Skoufranis

York University

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Convex Hulls of Unitary Orbits of Operators

Question

Given a unital C*-algebra $\mathfrak{A}$ and a self-adjoint operator $T \in \mathfrak{A}$, is it possible to describe

$$\overline{\mathrm{conv}}(\mathcal{U}(T)) = \overline{\mathrm{conv}(\{U^* T U \mid U \in \mathcal{U}(\mathfrak{A})\})}$$

using spectral data?

Elements of $\overline{\mathrm{conv}}(\mathcal{U}(T))$ are 'averages' of elements of $\mathcal{U}(T)$. Thus we think of T 'majorizing' elements of $\overline{\mathrm{conv}}(\mathcal{U}(T))$.

Dixmier Property

Definition

A unital C*-algebra $\mathfrak{A}$ is said to have the *Dixmier Property* if $\overline{\mathrm{conv}}(\mathcal{U}(T)) \cap Z(\mathfrak{A}) \neq \emptyset$ for all $T \in \mathfrak{A}$.

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Theorem (Archbold, Robert, Tikusis; 2017)

A unital C^*-algebra $\mathfrak{A}$ has the Dixmier property if and only if $\mathfrak{A}$ has the following properties:

- *$\mathfrak{A}$ is weakly central,*
- *every simple quotient of $\mathfrak{A}$ has at most one tracial state, and*
- *every extreme tracial state of $\mathfrak{A}$ factors through some simple quotient.*

C^*-Simplicity

Theorem (Haagerup; 2015), (Kennedy; 2015)

Let G be a discrete group. Then $C_r^*(G)$ is simple if and only if for all self-adjoint $T \in C_r^*(G)$ and for all $\epsilon > 0$ there exists $g_1, \ldots, g_n \in G$ such that

$$\left\| \tau(T) - \frac{1}{n} \sum_{k=1}^{n} \lambda(g_k)^* T \lambda(g_k) \right\| < \epsilon.$$

Quantum Channels

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- *mixed unitary quantum channel* if Ψ is a convex combination of unitary channels.

Hence mixed unitary quantum channels have the form $T \mapsto \sum_{k=1}^{m} t_k U_k^* T U_k$ where $t_k \in [0, 1]$ are such that $\sum_{k=1}^{m} t_k = 1$ and $U_1, \ldots, U_m$ are unitaries.

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Thus describing $\overline{\mathrm{conv}}(\mathcal{U}(T))$ describes the set of outputs of T under all mixed unitary quantum channels.

Characterizations of $\overline{\mathrm{conv}}(\mathcal{U}(B))$ - Matrix Algebras

Theorem (Various)

Let $A, B \in \mathcal{M}_n$ be self-adjoint with eigenvalues

$$a_1 \geq a_2 \geq \cdots \geq a_n \qquad \text{and} \qquad b_1 \geq b_2 \geq \cdots \geq b_n$$

respectively. The following are equivalent:

1. $A \in \mathrm{conv}(\mathcal{U}(B))$.
2. *there exists a double stochastic matrix $X \in \mathcal{M}_n$ such that $X\vec{b} = \vec{a}$.*
3. $\sum_{k=1}^{m} a_k \leq \sum_{k=1}^{m} b_k$ *for all $m \in \{1, \ldots, n\}$ with equality when $m = n$.*
4. $\mathrm{Tr}(f(A)) \leq \mathrm{Tr}(f(B))$ *for all $f : \mathbb{R} \to \mathbb{R}$ continuous and convex.*
5. *there exists a unital, trace-preserving, completely positive map $\Psi : \mathcal{M}_n \to \mathcal{M}_n$ such that $\Psi(B) = A$.*
6. *There exists unitaries $U, V \in \mathcal{M}_n$ such that $U^*AU = E_D(V^*BV)$ where $E_D : \mathcal{M}_n \to \mathcal{M}_n$ is the expectation onto the diagonal.*

Eigenvalue Functions in II_1 Factors

Let $\mathfrak{M}$ be a type II_1 factor with tracial state τ.

Definition

For a self-adjoint operator $T \in \mathfrak{M}$, the *eigenvalue function* of T is defined for $s \in [0, 1)$ by

$$\lambda_T^\tau(s) = \inf\{t \in \mathbb{R} \mid \mu_T((t, \infty)) \leq s\}.$$

For example, if $P_1, \ldots, P_n$ are pairwise orthogonal projections such that $\sum_{k=1}^n P_k = I_{\mathfrak{M}}$, if $a_1 > a_2 > \cdots > a_n$, if $T = \sum_{k=1}^n a_k P_k$, and if $s_k = \sum_{j=1}^k \tau(P_j)$, then

$$\lambda_T^\tau(s) = a_k \quad \text{for all } s \in [s_{k-1}, s_k).$$

Schur Horn Theorem - II_1 Factors

Let $\mathfrak{M}$ be a type II_1 factor with tracial state τ and let $T, S \in \mathfrak{M}$ be self-adjoint.

Theorem (Hiai, Nakamura; 1991)

The following are equivalent:

1. $S \in \overline{\mathrm{conv}}(\mathcal{U}(T))$.
2. $\int_0^y \lambda_S^\tau(x)\,dx \leq \int_0^y \lambda_T^\tau(x)\,dx$ *for all* $y \in [0,1]$ *with equality at* $y = 1$.
3. $\tau((S-\alpha)_+) \leq \tau((T-\alpha)_+)$ *and* $\tau((-S-\alpha)_+) \leq \tau((-T-\alpha)_+)$ *for all* $\alpha \in \mathbb{R}$.

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Theorem (Ravichandran; 2014)

$S \in \overline{\mathrm{conv}}(\mathcal{U}(T))$ *if and only if whenever* $\mathcal{A}$ *is a MASA of* $\mathfrak{M}$ *containing* S *and* $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ *is the conditional expectation onto* $\mathcal{A}$ *there exists a* $R \in \mathcal{U}(T)$ *such that* $E_{\mathcal{A}}(R) = S$.

Thompson's Theorems - II_1 Factors

Theorem (Kennedy, Skoufranis; 2017)

Let $\mathfrak{M}$ be a type II_1 factor with tracial state τ and let $T, S \in \mathfrak{M}$ with S normal. Then the following are equivalent:

1. $S \in \overline{\mathrm{conv}}(\{UTV \mid U, V \in \mathcal{U}(\mathfrak{M})\})$.
2. $\int_0^y \lambda^\tau_{|S|}(x)\,dx \leq \int_0^y \lambda^\tau_{|T|}(x)\,dx$ *for all* $y \in [0,1]$.
3. *If $\mathcal{A}$ is a MASA of $\mathfrak{M}$ containing S and $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ is the conditional expectation onto $\mathcal{A}$, then there exists a*

$$R \in \overline{\{UTV \mid U, V \in \mathcal{U}(\mathfrak{M})\}}$$

   such that $E_{\mathcal{A}}(R) = S$.

A Characterization of $\overline{\mathrm{conv}}(\mathcal{U}(T))$

Let $\mathfrak{A}$ be a unital C*-algebra and let $\mathcal{T}(\mathfrak{A})$ denote all 'unbounded traces'; that is, all maps $\tau : \mathfrak{A}_+ \to [0, \infty]$ such that

- $\tau(T + S) = \tau(T) + \tau(S)$ for all $T, S \in \mathfrak{A}_+$,
- $\tau(\alpha T) = \alpha\tau(T)$ for all $T \in \mathfrak{A}_+$ and $\alpha \in \mathbb{R}_+$ $(0 \cdot \infty = 0)$,
- $\tau(X^*X) = \tau(XX^*)$ for all $X \in \mathfrak{A}$, and
- τ is lower semicontinuous.

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Theorem (Ng, Robert, Skoufranis; 2018)

Let $\mathfrak{A}$ be a unital C^*-algebra and let $T, S \in \mathfrak{A}$ be self-adjoint. The following are equivalent:

- $S \in \overline{\mathrm{conv}}(\mathcal{U}(T))$.
- $\tau((S - \alpha)_+) \leq \tau((T - \alpha)_+)$ *and* $\tau((-S - \alpha)_+) \leq \tau((-T - \alpha)_+)$ *for all* $\tau \in \mathcal{T}(\mathfrak{A})$ *and* $\alpha \in \mathbb{R}$.

Convex Hulls of Joint Unitary Orbits

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Definition

Let $\mathfrak{A}$ be a unital C*-algebra. An m-tuple $\vec{A} = (A_1, \ldots, A_m) \in \mathfrak{A}^m$ is said to be an *abelian family* if A_k is self-adjoint and $A_k A_j = A_j A_k$ for all $1 \leq j, k \leq m$. The *(joint) unitary orbit of* $\vec{A}$ is

$$\mathcal{U}(\vec{A}) = \{(U^* A_1 U, \ldots, U^* A_m U) \mid U \in \mathfrak{A} \text{ is unitary}\} \subseteq \mathfrak{A}^m.$$

Thus

$$\mathrm{conv}(\mathcal{U}(\vec{A})) = \left\{ \sum_{i=1}^{k} t_i \vec{C}_i \,\middle|\, \substack{k \in \mathbb{N}, \{\vec{C}_i\}_{i=1}^k \subseteq \mathcal{U}(\vec{A}), \\ \vec{t} \in \mathbb{R}^k \text{ a probability vector}} \right\}.$$

Can $\overline{\mathrm{conv}}(\mathcal{U}(\vec{A}))$ be characterized using spectral data?

Joint Majorization - Tracial Majorization

Definition

Let $\mathfrak{A}$ be a unital C*-algebra and let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be abelian families in $\mathfrak{A}$.

It is said that $\vec{A}$ is *tracially majorized* by $\vec{B}$, denoted $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$, if for every tracial state $\tau : \mathfrak{A} \to \mathbb{C}$ and every continuous convex function $f : \mathbb{R}^m \to \mathbb{R}$ we have that

$$\tau(f(A_1, \ldots, A_m)) \leq \tau(f(B_1, \ldots, B_m)).$$

Joint Majorization - Doubly Stochastic Majorization

Definition

Let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be abelian families in $\mathcal{M}_n$. Using simultaneously diagonalizable, we can write

$$A_j = U^* D_j U \quad \text{and} \quad B_j = V^* D'_j V$$

for all $1 \leq j \leq m$ for some unitary matrices $U, V \in \mathcal{M}_n$ and diagonal matrices D_j and D'_j. Let A and B be the $n \times m$ matrices whose j^{th} columns are the diagonal entries of D_j and D'_j respectively.

It is said that $\vec{A}$ is *(doubly stochastic) majorized* by $\vec{B}$, denoted $\vec{A} \prec \vec{B}$, if there exists a $X \in DS_n$ such that $XB = A$.

Joint Majorization - Matrix Algebras

Theorem (Various)

Let $\vec{A} = (A_1, \ldots, A_m)$ *and* $\vec{B} = (B_1, \ldots, B_m)$ *be abelian families in* $\mathcal{M}_n$*. The following are equivalent:*

1. $\vec{A} \in \mathrm{conv}(\mathcal{U}(\vec{B}))$.
2. $\vec{A} \prec \vec{B}$.
3. $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$.
4. *There exists a unital, trace-preserving, (completely) positive map*

$$\Phi : C^*(A_1, \ldots, A_m) \to C^*(B_1, \ldots, B_m)$$

such that $\Phi(B_k) = A_k$ *for all* $1 \leq k \leq m$.

Joint Majorization - II_1 Factors

Theorem (Argerami, Massey; 2008)

Let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be abelian families in a II_1 factor $\mathfrak{M}$ with tracial state τ. The following are equivalent:

1. $\vec{A} \in \overline{\mathrm{conv}}(\mathcal{U}(\vec{B}))$.
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Joint Majorization - Partial C*-Algebra Results

Theorem (Hu, Lin; 2020)

Let $\mathfrak{A}$ be a unital, separable, simple, C*-algebra with tracial rank zero. For any normal operators $A, B \in \mathfrak{A}$, $A \in \overline{\mathrm{conv}}(\mathcal{U}(B))$ if and only if there exists a sequence of unital completely positive maps $\Psi_n : \mathfrak{A} \to \mathfrak{A}$ such that

- $\lim_{n\to\infty} \|\Psi_n(B) - A\| = 0$, *and*
- $\tau(\Psi_n(C)) = \tau(C)$ *for all* $\tau \in T(\mathfrak{A})$ *and* $C \in \mathfrak{A}$.

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Theorem (Mootoo, S; 2022)

Let $\mathfrak{A}$ be a unital C-algebra and let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be abelian families in $\mathfrak{A}$. If $\vec{A} \in \overline{\mathrm{conv}}(\mathcal{U}(\vec{B}))$, then $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$.*

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To what extent does the converse of the above theorem hold in an arbitrary C*-algebra?

Joint Majorization - Partial C*-Algebra Resultsn

Theorem (Mootoo, S; 2022)

Let $\mathcal{X}$ be a compact Hausdorff space and let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be abelian families in $C(\mathcal{X}, M_n)$. The following are equivalent:

1. $\vec{A} \prec_{\mathrm{pt}} \vec{B}$.
2. $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$.
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2. $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$.
3. $\vec{A} \in \overline{\mathrm{conv}}(\mathcal{U}(\vec{B}))$.

Theorem (Mootoo, S; 2022)

Let $\mathfrak{A}$ be a unital, separable, subhomogeneous C^*-subalgebra. Suppose $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ are abelian families in $\mathfrak{A}$. Then the following are equivalent:

1. $\vec{A} \prec_{\mathrm{Tr}} \vec{B}$.
2. $\vec{A} \in \overline{\mathrm{conv}}(\mathcal{U}(\vec{B}))$.

Convex Hulls of Joint Unitary Orbits - Non-Commutative

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Theorem (Helton, Klep, McCullough; 2017)

Given $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ in $\mathcal{M}_n$, there exists a unital, completely positive, trace-preserving map $\Psi : \mathcal{M}_n \to \mathcal{M}_n$ such that $\Psi(B_k) = A_k$ for all k if and only if a specific semidefinite programming problem has a solution.

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Theorem (Gour, Jennings, Buscemi, Duan, Marvian; 2018)

Given $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ in $\mathcal{M}_n$, $\vec{B}$ quantum majorizes $\vec{A}$ (i.e. there exists a completely positive, trace-preserving map $\Psi : \mathcal{M}_n \to \mathcal{M}_n$ such that $\Psi(B_k) = A_k$ for all k) if and only if specific entropy conditions hold.

Upcoming Results

Theorem (Kennedy, Marcoux, S; 2023)

Let $\mathfrak{A}$ be a C^*-algebra, let τ be a tracial state on $\mathfrak{A}$, and let $\vec{A} = (A_1, \ldots, A_m)$ and $\vec{B} = (B_1, \ldots, B_m)$ be m-tuples in $\mathfrak{A}$. There exists a [unital] trace-preserving, completely positive map

$$\Psi : C^*(B_1, \ldots, B_m) \to C^*(A_1, \ldots, A_m)$$

such that $\Psi(B_k) = A_k$ for all k if and only if $\vec{A} \prec_{nc} \vec{B}$.

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- $\vec{A} \prec_{nc} \vec{B}$ is a direct generalization of tracial majorization to the non-commutative context.
- In the non-abelian setting, majorization is about quantum channels; not mixed unitary quantum channels.

Thanks for Listening!