A new index theorem for manifolds with singularities and its geometric applications

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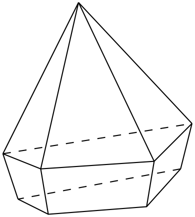
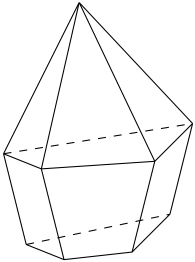
Stoker's conjecture on convex Euclidean polyhedra

Stoker's conjecture (1968)

If P_1 and P_2 are two convex Euclidean polyhedra of the same combinatorial type in $\mathbb{R}^3$. If the corresponding dihedral angles of P_1 and P_2 are equal, then the corresponding face angles of P_1 and P_2 are equal. Consequently, P_1 and P_2 are isometric up to translations of faces.





Some previously known results

1. **Karcher 1968**, a polyhedron with 5-vertices and 6 triangular faces.
2. **Andreev 1970**, an analogue of Stoker's conjecture for convex *hyperbolic* polyhedra when all dihedral angles are less than $\pi/2$.
3. **Schlenker 2000**, counterexamples to an analogue of Stoker's conjecture for convex *spherical* polyhedra
4. **Mazzeo and Montcouquiol 2011**, an infinitesimal (hence weaker) version of Stoker's conjecture
5. **Pogorelov 2002**, **Weiss 2005, 2009**, **Montcouquiol 2012**, ...

Positive solution to Stoker's conjecture

Theorem (Wang-X 2022)

If P_1 and P_2 are two convex Euclidean polyhedra of the same combinatorial type in $\mathbb{R}^n$. If the corresponding dihedral angles of P_1 and P_2 are equal, then the corresponding face angles of P_1 and P_2 are equal. Consequently, P_1 and P_2 are isometric up to translations of faces.

Gromov's dihedral extremality conjecture

Conjecture (Gromov's dihedral extremality conjecture for convex polyhedra)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a Riemannian metric on P such that

1. *(scalar curvature comparison)* $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. *(mean curvature comparison)* $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face* F_i *of* P, *and*
3. *(dihedral angle comparison)* $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each* $F_{ij} = F_i \cap F_j$,

then we have

$$\mathrm{Sc}(g) = 0, H_g(F_i) = 0 \text{ and } \theta_{ij}(g) = \theta_{ij}(g_0)$$

for all i *and all* $j \neq i$.

Gromov's dihedral rigidity conjecture

Conjecture (Gromov's dihedral rigidity conjecture for convex polyhedra)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face* F_i *of* P,
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each* $F_{ij} = F_i \cap F_j$,

then g is also a flat metric.

Gromov's dihedral extremality/rigidity conjectures are strengthenings of the positive mass theorem, a foundational result in general relativity and differential geometry by Schoen-Yau (1979) and Witten (1981).

Some previously known results

1. **Gromov 2014**, the dihedral *extremality* conjecture holds for the standard Euclidean cube.
2. **Li 2019**, the dihedral rigidity conjecture for some special convex polyhedra with dimension ≤ 7, under extra assumptions on dihedral angles and combinatorial types (e.g. dihedral angles have to be **non-obtuse** and the cobimatorial type needs to be a prism: a polyhedron of dimension ≤ 7 that is a direct product $P_0 \times [0,1]^{n-2}$, where $P_0 \subset \mathbb{R}^2$ is a 2-dimensional polygon with non-obtuse dihedral angles.)

Positive solutions to Gromov's dihedral conjectures

Theorem (Wang-X-Yu 2021)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ *for each face F_i of P, and*
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ *on each $F_{ij} = F_i \cap F_j$,*

then we have

$$\mathrm{Sc}(g) = 0, H_g(F_i) = 0 \text{ and } \theta_{ij}(g) = \theta_{ij}(g_0)$$

for all i and all $j \neq i$. Moreover, g is Ricci flat.

Gauss-Bonnet theorem

Theorem (Gauss-Bonnet)

Given an oriented compact surface X without boundary,

$$2\pi \cdot \chi(X) = \int_X \kappa(x) dA$$

where $\chi(X)$ is the Euler characteristic of X and $\kappa(x)$ is the scalar curvature of X.

Gauss-Bonnet Theorem for surfaces with boundary

Theorem (Gauss-Bonnet)

Suppose (X, g) is an oriented surface with piecewise smooth boundary. Then

$$2\pi \cdot \chi(X) = \int_X \kappa_g(x) dA + \sum \int_{C_i} H_g(s) ds - \sum \theta_i + \pi \cdot \#\text{vertices}$$

where θ_i is the dihedral angle between C_i and C_{i+1}.

A consequence of the Gauss-Bonnet theorem

Let (P, g_0) be a convex polygon in $\mathbb{R}^2$. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq 0$,
2. $H_g(C_i) \geq 0$ for each edge C_i of P, and
3. $\theta_i(g) \leq \theta_i(g_0)$ at each vertex $v_i = C_i \cap C_{i+1}$,

then $\mathrm{Sc}(g) = 0$, $H_g(C_i) = 0$, and $\theta_i(g) = \theta_i(g_0)$.

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$$2\pi \cdot \chi(X) = \int_X \kappa_g(x) dA + \sum \int_{C_i} H_g(s) ds - \sum \theta_i(g) + \pi \cdot \#\text{vertices}$$

$$2\pi \cdot \chi(X) = \quad 0 \quad + \quad 0 \quad - \sum \theta_i(g_0) + \pi \cdot \#\text{vertices}$$

Dirac operator on $\mathbb{R}^n$

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On $\mathbb{R}^1$,

$$\text{Laplacian } = -\frac{d^2}{dx^2}, \qquad D = -i\frac{d}{dx}$$

Dirac operator on $\mathbb{R}^n$

On $\mathbb{R}^2$, Laplacian $= -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

$$D = c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y}$$

where

$$c_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } c_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

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On $\mathbb{R}^n$, there is a similar construction.

To define the Dirac operator on $\mathbb{R}^2$, we have used the fact

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}.$$

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On a general manifold,

$$D^2 = \text{Laplacian} + \frac{\text{scalar curvature}}{4}$$

because partial derivatives **do not** commute in general, which gives rise to a curvature term.

Dirac operators on manifolds

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To define the Dirac operator on a general manifold, the manifold needs a certain orientability condition, called **spin**.

Dirac operators twisted by vector bundles

On a general spin manifold X, the Dirac operator D actually is defined on a spinor bundle S_X. Now suppose E is another vector bundle over X. Then one can define a twisted Dirac operator D_E on $S_X \otimes E$. In this case,

$$D_E^2 = \text{Laplacian} + \frac{\text{scalar curvature}}{4} + \mathcal{R}^E$$

where $\mathcal{R}^E$ is some extra term determined by the curvature of E.

Strategy of the proof

1. Use an appropriate twisted Dirac operator to obtain comparisons of scalar curvature, mean curvature and dihedral angles
2. develop the index theory for manifolds with polyhedral boundary and apply it to this twisted Dirac operator

Gromov's flat corner domination conjecture

Gromov's flat corner domination conjecture

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1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
2. $H_g(F_i) \geq H_{g_0}(F_i) = 0$ for each face F_i of P, and
3. $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ on each $F_{ij} = F_i \cap F_j$,

then g is also a flat metric, all codimension one faces of (P, g) are flat, and (P, g) and (P, g_0) are locally isometric.

Positive solution to Gromov's flat corner domination conjecture

Theorem (Wang-X 2022)

Let P be a convex polyhedron in $\mathbb{R}^n$ and g_0 the Euclidean metric on P. If g is a smooth Riemannian metric on P such that

1. $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_0) = 0$,
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An application of Gromov's flat corner domination conjecture

Theorem (Wang-X 2022)

If P_1 and P_2 are two convex Euclidean polyhedra of the same combinatorial type in $\mathbb{R}^n$. If the corresponding dihedral angles of P_1 and P_2 are equal, then the corresponding face angles of P_1 and P_2 are equal. Consequently, P_1 and P_2 are isometric up to translations of faces.

This answers positively the Stoker conjecture.

Comparison of scalar curvature, mean curvature and dihedral angles on polyhedra

Proposition (Wang-X-Yu 2021)

Let (P, g_0) be a convex Euclidean polyhedron. Let g be another Riemannian metric on P. Suppose D is the twisted Dirac operator on $S_{g_0} \otimes S_g$. Then we have

$$\int_P |D\varphi|^2 \geq \int_P |\nabla\varphi|^2 + \int_P \frac{\mathrm{Sc}_g - 0}{4}|\varphi|^2 + \sum_i \int_{F_i} \frac{H_g - 0}{2}|\varphi|^2 + \frac{1}{2}\sum_{i,j}\int_{F_i \cap F_j} (\theta_{ij}(g_0) - \theta_{ij}(g))|\varphi|^2 + \sum_i \int_{F_i} \langle D^\partial \varphi, \varphi \rangle$$

for all smooth sections φ of $S_{g_0} \otimes S_g$.

Here the bundle $S_{g_0} \otimes S_g$ is isomorphic (but generally not isometric) to the bundle of differential forms $\Lambda^\bullet T^* P$.

Need to find a nontrivial φ such that $D\varphi = 0$. Then

$$0 = \int_P |D\varphi|^2 \geq \int_P |\nabla\varphi|^2 + \int_P \frac{\mathrm{Sc}_g - 0}{4}|\varphi|^2 + \sum_i \int_{F_i} \frac{H_g - 0}{2}|\varphi|^2$$

$$+ \frac{1}{2}\sum_{i,j}\int_{F_i \cap F_j} (\theta_{ij}(g_0) - \theta_{ij}(g))|\varphi|^2 + \cancelto{0}{\sum_i \int_{F_i} \langle D^\partial \varphi, \varphi\rangle}$$

A new index theorem on manifolds with singularities

Theorem (Wang-X-Yu 2021)

Let (P, g_0) be a convex Euclidean polyhedron. Let g be another Riemannian metric on P. Suppose D is the twisted Dirac operator on $S_{g_0} \otimes S_g$ subject to the boundary condition $\mathbf{B}$ induced by $(\overline{\omega} \otimes \omega)(\overline{c}(\overline{e}_n) \otimes c(e_n))$. If the dihedral angles $\theta_{ij}(g)$ and $\theta_{ij}(g_0)$ satisfy

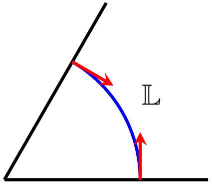
$$0 < \theta_{ij}(g)_z \leq \theta_{ij}(g_0)_z < \pi$$

for all codimension one faces $\overline{F}_i, \overline{F}_j$ of P and all $z \in \overline{F}_i \cap \overline{F}_j$, then $D_{\mathbf{B}}$ is an essentially self-adjoint Fredholm operator with Fredholm index

$$\mathrm{Ind}(D_{\mathbf{B}}) = \chi(P) = 1,$$

where $\chi(P)$ is Euler characteristic of P.

Essential self-adjointness of D_B



The de Rham operator (written in cylindrical coordinates) is

$$\begin{pmatrix} 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$$

where

$$P = \begin{pmatrix} -1/2 & -\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \theta} & -1/2 \end{pmatrix}$$

Essential self-adjointness of D_B

The boundary condition B for P is that

$$\phi_1(0) = \phi_1(\alpha) = 0 \text{ for } \phi = \phi_0(\theta) + \phi_1(\theta)d\theta.$$

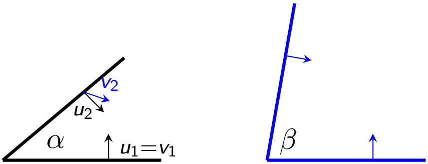
The spectrum of P subject to the boundary condition B is

$$\left\{ -\frac{1}{2} + \frac{k\pi}{\alpha} \right\}_{k\in\mathbb{Z}}.$$

Lemma (Cheeger, Chou, Brüning-Seeley, $\cdots$)

Let P_B be the operator P on the link subject to the induced boundary B. Assume that P_B is essentially self-adjoint. Then D_B^{dR} is essentially self-adjoint $\Leftrightarrow$ the deficiency indices of D_B^{dR} are zero $\Leftrightarrow$ $|P_B| \geq 1/2$. Here the deficiency indices of D_B^{dR} are codim Ran$(D_B^{dR} \pm i)$.

Essential self-adjointness of D_B



The boundary condition B for P in this case is: $\phi_1(0) = 0$ and

$$-\phi_0(\alpha)\sin(\frac{\beta-\alpha}{2}) + \phi_1(\alpha)\cos(\frac{\beta-\alpha}{2}) = 0$$

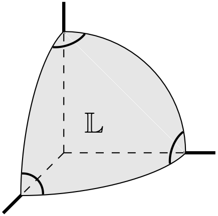
for $\phi = \phi_0(\theta) + \phi_1(\theta)d\theta$.

The spectrum of P_B is $\left\{ -\frac{\beta}{2\alpha} + \frac{k\pi}{\alpha} \right\}_{k\in\mathbb{Z}}$.

$|P_B| \geq 1/2$ if and only if $(\alpha + \beta \leq 2\pi$ and $\alpha \leq \beta)$.

Essential self-adjointness of D_B

The higher dimensional case is proved by induction. For example, near a singular point of codimension ℓ, the link is a polygon in $\mathbb{S}^{\ell-1}$.

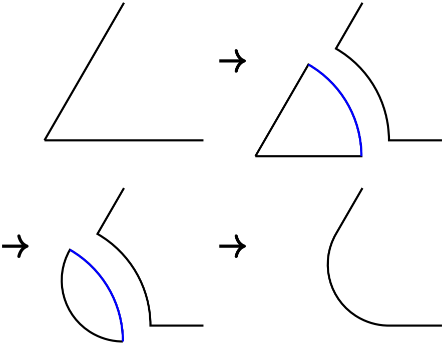


$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & 0 \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$$

In this case, we show that

$$|P_B|^2 \geq \frac{(\ell-1)(\ell-2)}{4}.$$

The Fredholm index is computed via a cutting-and-pasting argument together with a delicate deformation argument.



Thank you!