C*-algebras generated by isometries and their boundary quotients

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$P =$ submonoid of a group $G \quad (e \in P \subset G)$

Left regular representation $\quad p \mapsto L_p \quad$ of P by isometries on $\ell^2(P)$:

$$L_p \delta_q = \delta_{pq} \qquad p \in P,$$

defined first on $\{\delta_q : q \in P\}$, then extended by linearity and continuity.

The (reduced) Toeplitz C*-algebra is the C*-algebra generated by L

$$\mathcal{T}_\lambda(P) := C^*(L_p : p \in P) \subset \mathcal{B}(\ell^2(P)).$$

Spatial techniques are available to study $\mathcal{T}_\lambda(P)$, but estimating norms of operators is difficult, so it is not easy to decide whether a given collection $\{V_p\}_{p \in P}$ produces a representation of $\mathcal{T}_\lambda(P)$.

- We must send $L_p \mapsto V_p$ and $L_p^* \mapsto V_p^*$, and for each polynomial F on $2k$ noncommuting variables we need to send

$$F(L_{p_1}, \cdots, L_{p_k}; L_{p_1}^*, \cdots, L_{p_k}^*) \mapsto F(V_{p_1}, \cdots, V_{p_k}; V_{p_1}^*, \cdots, V_{p_k}^*)$$

but then we need to prove

$$\|F(V_{p_1}, \cdots, V_{p_k}; V_{p_1}^*, \cdots, V_{p_k}^*)\| \overset{?}{\leqslant} \|F(L_{p_1}, \cdots, L_{p_k}; L_{p_1}^*, \cdots, L_{p_k}^*)\|$$

which is not easily done.

- On the other hand, if a C*-algebra A is characterized in terms of generators and relations, then every collection of elements satisfying the defining relations automatically gives a representation of A.

- A strategy that is often successful is to come up with a suitable presentation that characterizes $\mathcal{T}_\lambda(P)$ as well as possible.

three classical theorems

- (Coburn '67) Let $S =$ unilateral shift and $V =$ an isometry. Then $S^n \mapsto V^n$ $(n \in \mathbb{N})$ extends to a homomorphism $C^*(S) \to C^*(V)$, which is an isomorphism iff $V V^* \neq 1$.

- (Douglas '72) Let Γ be a subgroup of $\mathbb{R}$ and let $\Gamma^+ := \Gamma \cap [0, \infty)$. Suppose L is the l.r.r. of Γ^+ and V is another representation of Γ^+. Then $L_p \mapsto V_p$ extends to a homomorphism $\mathcal{T}_\lambda(\Gamma^+) \longrightarrow C^*(V)$, which is an isomorphism iff $V_p V_p^* \neq 1$ for some (hence all) $p \neq 0$.

- (Cuntz '81) Suppose L is the l.r.r. and V is an isometric representation of $P = \mathbb{F}_n^+$, the free monoid on n generators $\{1, 2, \cdots n\}$, and assume $\sum_{j=1}^n V_j V_j^* \leqslant 1$. Then $L_p \mapsto V_p$ extends to a homomorphism $\mathcal{T}_\lambda(\mathbb{F}_n^+) \longrightarrow C^*(V)$, which is an isomorphism iff $\prod_{j=1}^n (1 - V_j V_j^*) \neq 0$.

presentations for $\mathcal{T}_\lambda(\mathbb{N})$, $\mathcal{T}_\lambda(\Gamma^+)$, and $\mathcal{T}_\lambda(\mathbb{F}_n^+)$

- Semigroup: Presentation
- $P = \mathbb{N}$; $v^* v = 1$,
- $P = \Gamma^+$; $v_\gamma^* v_\gamma = 1$, $v_\gamma v_\delta = v_{\gamma+\delta}$ for $\gamma, \delta \in \Gamma^+$
- $P = \mathbb{F}_n^+$; $v_j^* v_j = 1$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n v_j v_j^* \leqslant 1$

In each case:

1. $\mathcal{T}_\lambda(P)$ is universal (surprising for $\mathbb{F}_n^+$ because $\mathbb{F}_n$ is nonamenable)
2. $\mathcal{T}_\lambda(P)$ is unique for 'jointly proper' representations
3. $\exists$ boundary quotient $\partial\mathcal{T}_\lambda(P)$ for 'maximally improper' representations

more general semigroups

- [Nica '92]: A new class of semigroups: $P \subset G$ is *quasi-lattice ordered* if $P \cap P^{-1} = \{e\}$ and for $x, y \in G$,
  $$xP \cap yP = \begin{cases} zP & \text{when } xP \cap yP = zP \\ 0 & \text{when } xP \cap yP = \varnothing \end{cases}$$
  Presentation: consider isometric representations of P that satisfy
  $$(V_p V_p^*)(V_q V_q^*) = \begin{cases} V_z V_z^* & \text{when } xP \cap yP = zP \\ 0 & \text{when } xP \cap yP = \varnothing \end{cases}$$
  This led to semigroup crossed products [L-Raeburn] and to boundary relations that characterize boundary quotient [L-Crisp '07].

- [Li '12] Any $P \subset G$ with $e \in P$

  Presentation: consider isometric representations of P that preserve the semi-lattice structure of a distinguished collection of subsets of P (the constructible right ideals).

Xin Li's constructible right ideals: motivation by example

For $p, q, r, s \in P$ compute $L_p^* L_q L_r^* L_s$ acting on $\ell^2(P)$:

$$(L_p^* L_q L_r^* L_s)\delta_x = L_p^* L_q L_r^* \delta_{sx} = \begin{cases} L_p^* L_q \delta_{r^{-1}sx} & \text{if } sx \in rP (\Leftrightarrow x \in s^{-1}rP) \\ 0 & \text{otherwise.} \end{cases}$$

Assuming from now on $x \in s^{-1}rP$, we continue

$$L_p^* L_q \delta_{r^{-1}sx} = \begin{cases} \delta_{p^{-1}qr^{-1}sx} & \text{if } x \in s^{-1}rq^{-1}pP \\ 0 & \text{otherwise.} \end{cases}$$

So $$L_p^* L_q L_r^* L_s \delta_x = \begin{cases} \delta_{(p^{-1}qr^{-1}s)x} & \text{if } x \in P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP, \\ 0 & \text{otherwise.} \end{cases}$$

$K(p, q, r, s) := P \cap s^{-1}rP \cap s^{-1}rq^{-1}pP$ is a constructible right ideal.

If we assume that $\alpha = (p, q, r, s)$ is neutral, i.e. $p^{-1}qr^{-1}s = e$, we get a projection

$$L_p^* L_q L_r^* L_s = \mathbb{1}_{K(p,q,r,s)}$$

constructible right ideals

In general, for each $k \in \mathbb{N}$ and each word $\alpha = (p_1, p_2, \dots, p_{2k})$ we set

$$\dot{\alpha} := p_1^{-1} p_2 \cdots p_{2k-1}^{-1} p_{2k} \in G,$$

$$\tilde{\alpha} := (p_{2k}, p_{2k-1}, \dots p_2, p_1) \quad \text{and define}$$

$$K(\alpha) := P \cap (p_{2k}^{-1} p_{2k-1}) P \cap (p_{2k}^{-1} p_{2k-1} p_{2k-2}^{-1} p_{2k-3}) P \cap \cdots \cap (\dot{\tilde{\alpha}}) P,$$

where $\dot{\tilde{\alpha}} := p_{2k}^{-1} p_{2k-1} \cdots p_2^{-1} p_1$.

Let $\mathcal{W} = \{$ words of even length over $P\}$

The collection of constructible right ideals

$$\mathcal{J} = \{K(\alpha) : \alpha \in \mathcal{W}\} = \{K(\alpha) : \alpha \in \mathcal{W},\ \dot{\alpha} = e\}$$

is a semi-lattice under intersection.

universal Toeplitz C*-algebra $\mathcal{T}_u(P)$

Definition [L-Sehnem] Let $\mathcal{T}_u(P)$ be the universal C*-algebra with generators $\{t_p \colon p \in P\}$ such that (with $\dot{t}_\alpha := t^*_{p_1} t_{p_2} \cdots t^*_{p_{2k-1}} t_{p_{2k}}$)

(T1) $t_e = 1$;

(T2) $\dot{t}_\alpha = 0$ if $K(\alpha) = \emptyset$ with $\dot{\alpha} = e$;

(T3) $\dot{t}_\alpha - \dot{t}_\beta = 0$ if $K(\alpha) = K(\beta)$ for α and β such that $\dot{\alpha} = e = \dot{\beta}$;

(T4) $\prod_{\beta \in F}(\dot{t}_\alpha - \dot{t}_\beta) = 0$ if $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$ for some α and finite set F with $\dot{\alpha} = e = \dot{\beta}$.

$$\mathcal{T}_u(P) := C^*(\{t_p : p \in P\}) = \overline{\operatorname{span}}\{\dot{t}_\alpha : \alpha \in \mathcal{W}\}$$

$$D_u := C^*(\{\dot{t}_\alpha \dot{t}^*_\alpha : \alpha \in \mathcal{W}\}) = \overline{\operatorname{span}}\{\dot{t}_\alpha : \alpha \in \mathcal{W},\ \dot{\alpha} = e\}$$

some consequences

Relations (T1)–(T3) give a presentation of Li's $C_s^*(P)$: hence $\{t_p : p \in P\}$ is a semigroup of isometries generating $\mathcal{T}_u(P)$ and $\{\dot{t}_\alpha : \dot{\alpha} = e\}$ is a commuting family of projections.

Moreover,

$$C_s^*(P) \xrightarrow{\pi} \mathcal{T}_u(P) \xrightarrow{\pi_L} \mathcal{T}_\lambda(P)$$

$$D_s \xrightarrow{\pi|_D} D_u \xrightarrow{\cong} D_\lambda$$

π_L is an isomorphism iff the cond. expect. $E_u : \mathcal{T}_u(P) \to D_u$ is faithful, e.g. for amenable G, (but also for many nonamenable G).

whether π and $\pi|_D$ are isomorphisms depends on independence.

independence: what it is and how it can fail

P satisfies independence iff any one of the following holds:

- $K(\alpha) = \bigcup_{\beta \in F} K(\beta) \implies K(\beta) = K(\alpha)$ for some $\beta \in F$.
- $\{\mathbb{1}_{K(\alpha)} : K(\alpha) \in \mathcal{J}\}$ is linearly independent in $\ell^\infty(P)$.
- $D_s \xrightarrow{\pi|_D} D_u (= D_\lambda)$ is an isomorphism
- $C_s^*(P) \xrightarrow{\pi} \mathcal{T}_u(P)$ is an isomorphism

Failures of independence:

Example 1 (Li '17) Independence fails on $\Sigma = \{0, 2, 3, \ldots\} \subset \mathbb{Z}$ because

$$K(3, 2, 2, 3) = 2 + \mathbb{N} \text{ can be written as } (2 + \Sigma) \cup (3 + \Sigma).$$

Example 2 (L-Sehnem) Independence fails for all multiplicative monoids, and all $ax + b$ monoids, of nonmaximal orders $\mathcal{O}$ in number fields.

$\mathcal{O}$ = any free full-rank proper subring of the ring $\mathcal{O}_K$ of integers in an algebraic number field K.

a partial action $G \circlearrowright D_\lambda$

There is a partial action γ of G on D_λ such that for $p \in P$,

$$\gamma_p(\mathbb{1}_{K(\alpha)}) = \gamma_p(\dot{L}_\alpha) = \dot{L}_{(e,p,\alpha,p,e)} = L_p \dot{L}_\alpha L_p^* = \mathbb{1}_{pK(\alpha)},$$

[X. Li]: $\mathcal{T}_\lambda(P) \;\cong\; D_\lambda \rtimes_{\gamma,r} G$

[L-Sehnem '21]: $\mathcal{T}_u(P) \;\cong\; D_u \rtimes_\gamma G.$

This gives

$$D_u \rtimes_u G \cong \mathcal{T}_u(P) \xrightarrow{\pi_L} \mathcal{T}_\lambda(P) \cong D_\lambda \rtimes_r G$$

faithful representations $\pi : \mathcal{T}_\lambda(P) \to \mathcal{B}(\mathcal{H})$

Let $P^* := P \cap P^{-1}$, the group of units in P.

Theorem [Li '17]: When $P^* = \{e\}$, π is faithful iff $\pi|_{D_\lambda}$ is faithful.

When $P^* \neq \{e\}$ we should not expect this to be true (take $P = G$).

The partial action $G \circlearrowright D_\lambda$ restricts to an action $P^* \circlearrowright D_\lambda$ and

$$\overline{\text{span}}\{\dot{t}_\alpha : \dot{\alpha} \in P^*\} \cong D_\lambda \rtimes_{\gamma,r} P^* \quad \hookrightarrow \quad D_\lambda \rtimes_{\gamma,r} G \;\cong\; \mathcal{T}_\lambda(P)$$

Theorem [L-Sehnem '21]: Every nontrivial ideal of $\mathcal{T}_\lambda(P)$ has nontrivial intersection with the subalgebra $D_\lambda \rtimes_{\gamma,r} P^*$.

(π is faithful on $\mathcal{T}_\lambda(P)$ iff it is faithful on $D_\lambda \rtimes_{\gamma,r} P^*$)

universality/uniqueness for $\mathcal{T}_\lambda(P)$

Under weak containment we get a general universality/uniqueness.

Theorem [L-Sehnem]: Suppose $E_u : \mathcal{T}_u(P) \to D_u$ is faithful, and let $\{W_p : p \in P\}$ be a collection of elements satisfying (T1)–(T4). Then $L_p \mapsto W_p$ extends to a homomorphism

$$\mathcal{T}_\lambda(P) \longrightarrow C^*(W),$$

which is injective iff its restriction to $D_\lambda \rtimes_{\gamma,r} P^*$ is injective.

topological freeness and jointly proper isometries

For some monoids it is possible to decide faithfulness/uniqueness based solely on the restriction to D_λ. The key is a property (TF) of P that ensures topological freeness of the partial action $P^* \circlearrowright D_\lambda$.

Definition: P satisfies (TF) if for every $u \in P^*\backslash\{e\}$ and

every $\mathcal{C} \subset\subset \mathcal{J}\backslash\{P\}$, there exists $q \in P\backslash\bigcup_{R\in\mathcal{C}} R$ such that $uqP \neq qP$.

Definition: $\{W_p : p \in P\}$ is *jointly proper* if $\prod_{\alpha\in F}(I - \dot{W}_\alpha) \neq 0$ for every finite collection F of neutral words with $K(\alpha) \neq P$.

Corollary [L-Sehnem '21]: Suppose $E_u : \mathcal{T}_u(P) \to D_u$ is faithful, P satisfies (TF), and $\{W_p : p \in P\}$ satisfies (T1)–(T4). Then $L_p \mapsto W_p$ extends to homomorphism

$$\mathcal{T}_\lambda(P) \longrightarrow C^*(W),$$

which is an isomorphism if and only if W is jointly proper.

boundary quotient from covariance algebras

Theorem [Li '17] (cf. L- Crisp '07) The spectrum $\Omega_P := \operatorname{Spec} D_\lambda$ has a smallest nonempty closed G-invariant subset $\partial\Omega_P$, and the reduced boundary is

$$\partial\mathcal{T}_\lambda(P) \;\cong\; C(\partial\Omega_P) \rtimes_r G.$$

By analogy, there is a full boundary,

$$\partial\mathcal{T}_u(P) \cong C(\partial\Omega_P) \rtimes_u G$$

We would like to have a presentation for $\partial\mathcal{T}_u(P)$.

The extra relations are derived from Sehnem's covariance algebra for product systems, in the particular case of the canonical product system with one-dimensional fibres associated to P.

foundation sets from Sehnem's strong covariance ideal

Definition: A *foundation set for the constructible right ideal* $K(\alpha)$ is a finite collection $\{K(\beta) : \beta \in F\} \subset \mathcal{J}$ such that

$$K(\alpha) \supset \bigcup_{\beta \in F} K(\beta) \quad \text{and} \quad pP \cap \bigcup_F K(\beta) \neq \emptyset \text{ for all } p \in K(\alpha).$$

((the $K(\beta)$s cover the shadow cones from $K(\alpha)$ 'at infinity'))

The foundation set $\{K(\beta) : \beta \in F\}$ is *proper* if $K(\alpha) \underset{\neq}{\supset} \bigcup_{\beta \in F} K(\beta)$.

this leads to the boundary relations

(T5) $\prod_{\beta \in F} (\dot{w}_\alpha - \dot{w}_\beta) = 0$ whenever $K(\alpha) \supset \bigcup_{\beta \in F} K(\beta)$ and

$pP \cap \bigcup_F K(\beta) \neq \emptyset$ for all $p \in K(\alpha)$.

We may assume proper inclusion, otherwise included in:

(T4) $\prod_{\beta \in F} (\dot{t}_\alpha - \dot{t}_\beta) = 0$ if $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$ for α and $F \underset{\text{fin}}{\subset} P$.

We may assume not indep., i.e. $K(\alpha) \neq K(\beta)\ \forall \beta \in F$, otherwise:

(T3) $\dot{t}_\alpha - \dot{t}_\beta = 0$ if $K(\alpha) = K(\beta)$ for α and β.

the full boundary quotient: *"there is no (T6)"*

Lemma [L-Sehnem] (T1)–(T5) is a maximal set of relations, i.e. the quotient of $\mathcal{T}_u(P)$ by any extra relation 'of the same kind' is trivial.

Proof: If $K(\alpha) \supset \bigcup_{\beta \in F} K(\beta)$ is not a foundation set, then $pP \cap \bigcup_{\beta \in F} K(\beta) = \varnothing$ for some $p \in K(\alpha)$, so $t_p t_p^* \leqslant \prod_{\beta \in F}(\dot{t}_\alpha - \dot{t}_\beta)$. If the product vanishes, then so does the isometry t_p. □

Theorem [L-Sehnem] The following are canonically isomorphic:

1. the covar. alg. $\mathbb{C} \rtimes_{\mathbb{C}^P} P$ of the 1-dim'l product system over P;
2. the universal C*-algebra with presentation (T1)–(T5);
3. the full partial crossed product $C(\partial\Omega_P) \rtimes_u G$.

In view of this we view the C*-algebra characterized in the theorem as the full boundary quotient of $\mathcal{T}_u(P)$.

purely infinite simple reduced boundary quotients

Theorem [L-Sehnem]: TFAE

1. Every proper ideal of $\partial\mathcal{T}_u(P)$ is contained in the kernel of the canonical map
$$\partial\mathcal{T}_u(P) \to \partial\mathcal{T}_\lambda(P) = C(\partial\Omega_P) \rtimes_r G.$$
2. The partial action $G \circlearrowright \partial\Omega_P$ is topologically free.
3. (∂TF) For all $p \neq q$ in P $\exists s \in P$ such that $psP \cap qsP = \emptyset$.

Corollary [L-Sehnem]: Assume $P \neq \{e\}$.

1) If condition (∂TF) above holds, then $\partial\mathcal{T}_\lambda(P)$ is purely infinite simple.

2) The converse holds if the boundary action satisfies weak containment (i.e. $\partial\mathcal{T}_u(P) \cong \partial\mathcal{T}_\lambda(P)$ via the canonical map).

Application: purely infinite C*-algebras from integral domains

Let R be an integral domain that is *not* a field and let $R \rtimes R^{\times}$ be the associated $b + ax$ monoid. So the multiplication is

$$(b, a)(d, c) = (b + ad, ac), \qquad b, d \in R, \ a, c \in R^{\times}.$$

Theorem [Cuntz '08, Li '10]: $\partial\mathcal{T}_{\lambda}(R \rtimes R^{\times})$ is purely infinite simple.

We can recover this by verifying directly that $P = R \times R^{\times}$ satisfies

(∂TF): $\forall p \neq q$ in P $\exists s \in P$ such that $psP \cap qsP = \emptyset$

Let $p = (b, a)$ and $q = (d, c)$ with $p \neq q$. We may assume $b \neq d$.

Case 1: $b - d \notin acR$. Set $s := (0, ac)$. Then $psP \cap qsP = \emptyset$ because, otherwise, $b - d \in acR$, contradicting the assumption.

Case 2: $b - d \in acR^{\times}$. Let $\bar{x} \in R^{\times}$ with $b - d = ac\bar{x}$. Let $r \in R^{\times}$ non-invertible and set $s := (0, ac\bar{x}r)$. Then $psP \cap qsP = \emptyset$ because, otherwise, r would be invertible, contradicting the assumption.

Application: uniqueness for C*-algebras of orders

Definition: Let K be a number field of degree d and let $\mathcal{O}_K$ be the ring of integers of K (it is a $\mathbb{Z}$-module of rank d). An *order* in K is a subring $\mathcal{O} \subset \mathcal{O}_K$ that is free of full rank as a $\mathbb{Z}$-module.

Example [Li-Norling '16]: Let $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$ (this is a proper subring of $\mathcal{O}_K$ for $K = \mathbb{Q}[\sqrt{-3}]$) Then the monoids $\mathcal{O}^\times$ and $\mathcal{O} \rtimes \mathcal{O}^\times$ do not satisfy independence.

Proposition [L-Sehnem]: For every nonmaximal order $\mathcal{O}$ in a number field, the monoids $\mathcal{O}^\times$ and $\mathcal{O} \rtimes \mathcal{O}^\times$ do not satisfy independence.

Theorem [L-Sehnem]: For every order in a number field, $\mathcal{T}_\lambda(\mathcal{O} \rtimes \mathcal{O}^\times)$ is universal and unique for jointly proper isometric representations satisfying (T1)-(T4).

Proof: The monoid $\mathcal{O} \rtimes \mathcal{O}^\times$ satisfies (TF).

Thank you!

Reference:

M. Laca and C. F. Sehnem, *Toeplitz algebras of semigroups*, Trans. Amer. Math Soc. **375** No. 10 (2022), 7443–7507.