Higman-Thompson groups from self-similar groupoid actions

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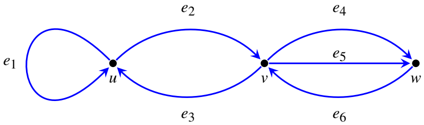
Outline

- We recall the concept of a self-similar groupoid action (G, E) on the path space of a finite graph.
- We describe when the corresponding ample groupoid of germs $\mathcal{G}(G, E)$ is Hausdorff, minimal, effective and purely infinite.
- Inspired by the work of Nekrashevych, we define the Higman-Thompson group $V_E(G)$ associated to (G, E) using G-tables and relate it to the topological full group $[\![\mathcal{G}(G, E)]\!]$.
- After recalling some concepts in groupoid homology, we discuss the Matui's AH-conjecture for $\mathcal{G}(G, E)$.
- We give a particular example of an element that is not in the kernel of the index map $I : [\![\mathcal{G}(G, E)]\!] \to H_1(\mathcal{G}(G, E))$.

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with no sources.

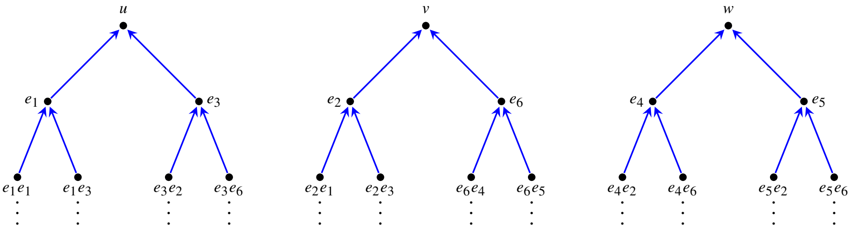


- The set of paths of length k is

$$E^k = \{e_1 e_2 \cdots e_k : e_i \in E^1,\ r(e_{i+1}) = s(e_i)\}.$$

- The space of finite paths is $E^* := \bigcup_{k \geq 0} E^k$ and E^∞ is the infinite path space with the topology given by $Z(\alpha) = \{\alpha\xi : \xi \in E^\infty\}$ for $\alpha \in E^*$.
- The set E^* is indexing the vertices of a forest T_E, where the level n has $|E^n|$ vertices.

- The forest T_E looks like



- A partial isomorphism of T_E is given by a bijection $g : u_1E^* \to u_2E^*$ preserving length and such that $g \cdot (\alpha e) \in (g \cdot \alpha)E^1$ for $\alpha \in E^k$ and $e \in E^1$.
- The set $\text{PIso}(T_E)$ forms a discrete groupoid with unit space E^0.
- In this example, $\text{PIso}(T_E)$ is transitive, but it could happen that there is no bijection $g : u_1E^* \to u_2E^*$ for $u_1 \neq u_2$.
- $\text{PIso}(T_E)$ could be a group bundle. If $|E^0| = 1$, then $\text{PIso}(T_E) = \text{Aut}(T_E)$ is a group.

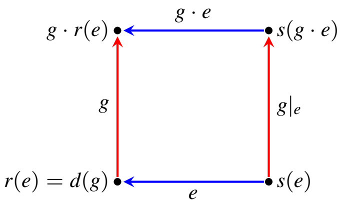
Self-similar groupoid actions

- Let E be a finite directed graph with no sources, and let G be a groupoid with unit space E^0. We denote by d and t the domain and target maps $G \to E^0$.
- **Definition**. A *self-similar action* (G, E) on the path space of E is given by a faithful groupoid homomorphism $G \to \mathrm{PIso}(T_E)$ such that for every $g \in G$ and every $e \in d(g)E^1$ there exists a unique $h \in G$ denoted by $g|_e$ and called the restriction of g to e such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \text{ for all } \mu \in s(e)E^*.$$

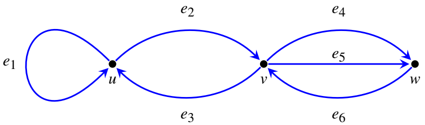
- We have

$$d(g|_e) = s(e),\ t(g|_e) = s(g \cdot e) = g|_e \cdot s(e),\ r(g \cdot e) = g \cdot r(e).$$



In general $s(g \cdot e) \neq g \cdot s(e)$, i.e. the source map is not G-equivariant.
- A self-similar action (G, E) is said to be level transitive if it is transitive on each E^n. The action is level transitive iff its extension to $\partial T_E = E^\infty$ is minimal.

- **Example**. Let E be the graph



- Consider the groupoid $G = \langle a, b, c\rangle$ and define the self-similar action (G, E) given by

$$a \cdot e_1 = e_2,\ \ a|_{e_1} = u,\ \ a \cdot e_3 = e_6,\ \ a|_{e_3} = b,$$

$$b \cdot e_2 = e_5,\ \ b|_{e_2} = a,\ \ b \cdot e_6 = e_4,\ \ b|_{e_6} = c,$$

$$c \cdot e_4 = e_2,\ \ c|_{e_4} = a^{-1},\ \ c \cdot e_5 = e_6,\ \ c|_{e_5} = b.$$

- Then for example

$$b \cdot e_2 e_1 = e_5(b|_{e_2} \cdot e_1) = e_5(a \cdot e_1) = e_5 e_2.$$

- Note that the action of G is level transitive. It can be shown that G is a transitive groupoid with isotropy $\mathbb{Z} = \langle a^{-1}cba\rangle$.

The groupoid $\mathcal{G}(G, E)$

- **Theorem**. If (G, E) is pseudo free ($g \cdot e = e$ and $g|_e = s(e)$ implies $g = r(e)$), then there is a locally compact Hausdorff étale groupoid of germs

$$\mathcal{G}(G, E) = \{[\alpha, g, \beta; \xi] : \alpha, \beta \in E^*,\ g \in G_{s(\beta)}^{s(\alpha)},\ \xi \in \beta E^\infty\},$$

  where $[\alpha, g, \beta; \beta\gamma\eta] = [\alpha(g \cdot \gamma), g|_\gamma, \beta\gamma; \beta\gamma\eta]$.
- The unit space of $\mathcal{G}(G, E)$ is identified with E^∞ by the map $[\alpha, s(\alpha), \alpha; \xi] \mapsto \xi$.
- The topology on $\mathcal{G}(G, E)$ is generated by the compact open bisections of the form

$$Z(\alpha, g, \beta) = \{[\alpha, g, \beta; \xi] \in \mathcal{G}(G, E) : \xi \in Z(\beta)\}.$$

- If G is amenable, then $\mathcal{G}(G, E)$ is also amenable and $C^*(G, E) = C^*(\mathcal{G}(G, E))$ is nuclear.
- $\mathcal{G}(G, E)$ is minimal iff E is G-transitive.
- $\mathcal{G}(G, E)$ is effective (essentially principal) iff
  (a) every G-circuit (a pair (g, α) with $s(\alpha) = g \cdot r(\alpha)$) has an entry;
  (b) for every $g \in G \setminus G^{(0)}$ there is $\zeta \in Z(d(g))$ such that $g \cdot \zeta \neq \zeta$.
- If $\mathcal{G}(G, E)$ is effective and minimal, then $\mathcal{G}(G, E)$ is purely infinite since it contains the graph groupoid $\mathcal{G}_E$,

$$\mathcal{G}_E = \{(\alpha\xi, |\alpha| - |\beta|, \beta\xi) \in E^\infty \times \mathbb{Z} \times E^\infty : \alpha, \beta \in E^*,\ s(\alpha) = s(\beta)\} \cong$$

$$\cong \{[\alpha, s(\alpha), \beta; \beta\xi] \in \mathcal{G}(G, E) : \alpha, \beta \in E^*,\ s(\alpha) = s(\beta),\ \xi \in E^\infty\}.$$

G-tables and the Higman-Thompson groups

- A G-table for (G,E) with $|uE^1|$ constant is a matrix of the form

$$\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ g_1 & g_2 & \cdots & g_m \\ \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix},$$

  where $\alpha_i, \beta_i \in E^*, g_i \in G_{s(\beta_i)}^{s(\alpha_i)}$ and $E^\infty = \bigsqcup_{i=1}^m Z(\alpha_i) = \bigsqcup_{i=1}^m Z(\beta_i)$.
- A G-table τ determines a homeomorphism $\bar{\tau}$ of E^∞ taking $\beta_i\xi$ into $\alpha_i(g_i \cdot \xi)$.
- The set of all such homeomorphisms is a countable subgroup $V_E(G)$ of $\mathrm{Homeo}(E^\infty)$, called the Higman-Thompson group.
- Given a G-table $\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ g_1 & g_2 & \cdots & g_m \\ \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix}$, the map

$$\tau \mapsto T = S_{\alpha_1} U_{g_1} S_{\beta_1}^* + S_{\alpha_2} U_{g_2} S_{\beta_2}^* + \cdots + S_{\alpha_m} U_{g_m} S_{\beta_m}^*$$

  defines a faithful unitary representation of the group $V_E(G)$ in the C^*-algebra $C^*(G,E)$.
- The Cuntz-Pimsner algebra $C^*(G,E)$ is generated by U_g, P_v and S_e such that
  - $g \mapsto U_g$ is a representation of G with $U_v = P_v$ for $v \in E^0$;
  - S_e are partial isometries with $S_e^* S_e = P_{s(e)}$ and $\sum_{r(e)=v} S_e S_e^* = P_v$;
  - $U_g S_e = \begin{cases} S_{g\cdot e} U_{g|_e} & \text{if } d(g) = r(e) \\ 0, & \text{otherwise;} \end{cases}$ $\quad U_g P_v = \begin{cases} P_{g\cdot v} U_g & \text{if } d(g) = v \\ 0, & \text{otherwise.} \end{cases}$

- If $\pi : X \to Y$ is a local homeomorphism between locally compact Hausdorff spaces, then for $f \in C_c(X, \mathbb{Z})$ define
$$\pi_*(f)(y) := \sum_{\pi(x)=y} f(x).$$
- It follows that $\pi_*(f) \in C_c(Y, \mathbb{Z})$.
- Given an étale groupoid $\mathcal{G}$ with domain and target maps $d, t : \mathcal{G} \to \mathcal{G}^{(0)}$, let $\mathcal{G}^{(1)} = \mathcal{G}$ and for $n \geq 2$ let $\mathcal{G}^{(n)}$ be the space of composable strings of n elements in $\mathcal{G}$ with the product topology.
- For $n \geq 2$ and $i = 0, \ldots, n$, we let $\partial_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ be the face maps defined by
$$\partial_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n-1, \\ (g_1, g_2, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$
- We define the homomorphisms $\delta_n : C_c(\mathcal{G}^{(n)}, \mathbb{Z}) \to C_c(\mathcal{G}^{(n-1)}, \mathbb{Z})$ given by
$$\delta_1 = d_* - t_*, \quad \delta_n = \sum_{i=0}^{n} (-1)^i \partial_{i*} \text{ for } n \geq 2.$$

Moerdijk-Crainic Homology

- It can be verified that $\delta_n \circ \delta_{n+1} = 0$ for all $n \geq 1$.
- The Moerdijk-Crainic homology groups $H_n(\mathcal{G}) = H_n(\mathcal{G}, \mathbb{Z})$ are by definition the homology groups of the chain complex $C_c(\mathcal{G}^{(*)}, \mathbb{Z})$ given by

$$0 \xleftarrow{\delta_0} C_c(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \longleftarrow \cdots,$$

  i.e. $H_n(\mathcal{G}) = \ker \delta_n / \mathrm{im}\, \delta_{n+1}$, where $\delta_0 = 0$.
- **Example**. For the action groupoid $\Gamma \ltimes X$ associated to a countable discrete group action $\Gamma \curvearrowright X$ on a Cantor set, it follows that

$$H_n(\Gamma \ltimes X) \cong H_n(\Gamma, C(X, \mathbb{Z})).$$

- Two equivalent groupoids have the same homology.
- **Theorem** (Ortega). For $\mathcal{G}$ an ample Hausdorff groupoid and $\rho : \mathcal{G} \to \mathbb{Z}$ a cocycle, we have the following long exact sequence

$$0 \longleftarrow H_0(\mathcal{G}) \longleftarrow H_0(\mathcal{G} \times_\rho \mathbb{Z}) \xleftarrow{id-\rho_*} H_0(\mathcal{G} \times_\rho \mathbb{Z}) \longleftarrow H_1(\mathcal{G}) \longleftarrow \cdots$$

$$\cdots \longleftarrow H_n(\mathcal{G}) \longleftarrow H_n(\mathcal{G} \times_\rho \mathbb{Z}) \xleftarrow{id-\rho_*} H_n(\mathcal{G} \times_\rho \mathbb{Z}) \longleftarrow H_{n+1}(\mathcal{G}) \longleftarrow \cdots$$

  where ρ_* is the map induced by the action $\hat{\rho} : \mathbb{Z} \curvearrowright \mathcal{G} \times_\rho \mathbb{Z}$.

- The study of full groups in the setting of topological dynamics was initiated by T. Giordano, I. F. Putnam and C. F. Skau.
- Topological full groups associated to dynamical systems and to étale groupoids are complete invariants for continuous orbit equivalence and for groupoid isomorphism.
- They provide means of constructing new groups with interesting properties, most notably by providing the first examples of finitely generated infinite simple groups that are amenable.
- The topological full group of an effective étale groupoid $\mathcal{G}$ is

$$[\![\mathcal{G}]\!] := \{\pi_U \mid U \subseteq \mathcal{G} \text{ full bisection}\},$$

  where $\pi_U := t|_U \circ (d|_U)^{-1}$ from $d(U) = \mathcal{G}^{(0)}$ to $t(U) = \mathcal{G}^{(0)}$, which is a subgroup of $\mathrm{Homeo}(\mathcal{G}^{(0)})$.
- **Theorem**. For a self-similar action (G, E) such that $\mathcal{G}(G, E)$ is effective, we have $V_E(G) \cong [\![\mathcal{G}(G, E)]\!]$. In particular, $V_E \cong [\![\mathcal{G}_E]\!] \subseteq [\![\mathcal{G}(G, E)]\!]$.
- A table $\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ g_1 & g_2 & \cdots & g_m \\ \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix}$ determines a full bisection $\bigsqcup_{i=1}^{m} Z(\alpha_i, g_i, \beta_i)$.

- The commutator subgroup of $[\![\mathcal{G}]\!]$ is denoted by $D([\![\mathcal{G}]\!])$ and the abelianization of $[\![\mathcal{G}]\!]$ is $[\![\mathcal{G}]\!]_{ab} = [\![\mathcal{G}]\!]/D([\![\mathcal{G}]\!])$.
- The AH-conjecture of Matui claims that for $\mathcal{G}$ effective minimal étale with $\mathcal{G}^{(0)}$ the Cantor set, the following sequence is exact

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} [\![\mathcal{G}]\!]_{ab} \xrightarrow{I_{ab}} H_1(\mathcal{G}) \to 0.$$

- The map I_{ab} is induced by the index map $I : [\![\mathcal{G}]\!] \to H_1(\mathcal{G})$ given by $\pi_U \mapsto [\chi_U]$, where $\chi_U \in \ker \delta_1 = \ker(d_* - t_*)$.
- The map j takes $[\chi_{d(U)}] \otimes \bar{1}$ into $[\pi_{\hat{U}}]$, where $U \subseteq \mathcal{G}$ is a bisection with $d(U) \cap t(U) = \emptyset$ and $\pi_{\hat{U}} \in [\![\mathcal{G}]\!]$ is the transposition corresponding to the full bisection

$$\hat{U} := U \sqcup U^{-1} \sqcup (\mathcal{G}^{(0)} \setminus (t(U) \cup d(U))).$$

- Matui proved that the AH-conjecture holds for the graph groupoid $\mathcal{G}_E$ and for other groupoids.
- If $\mathcal{T}(\mathcal{G})$ denotes the subgroup of $[\![\mathcal{G}]\!]$ generated by all transpositions, then $\mathcal{T}(\mathcal{G}) \subseteq \ker(I)$.
- We say that $\mathcal{G}$ has property TR if $\mathcal{T}(\mathcal{G}) = \ker(I)$. In order to verify the AH-conjecture for $\mathcal{G}$, it suffices to establish property TR.
- The initial motivation was to check the AH-conjecture for $\mathcal{G}(G,E)$ in particular cases and to understand the index map $I : [\![\mathcal{G}(G,E)]\!] \to H_1(\mathcal{G}(G,E))$.

Topological full groups and the AH-conjecture

- Using the new results of Xin Li, the AH-conjecture holds for all groupoids $\mathcal{G}(G,E)$, as long as they are purely infinite, minimal and ample. In particular, $\mathcal{G}(G,E)$ has the property TR.
- **Example**. The groupoid $\mathcal{G}(G,E)$ obtained from the self-similar action in the Example on page **??** has $H_1(\mathcal{G}(G,E)) \cong \mathbb{Z}$, so $I : [\![\mathcal{G}(G,E)]\!] \to \mathbb{Z}$.
- Consider the G-table $\tau = \begin{pmatrix} u & v & e_4 & e_5 \\ a^{-1} & cb & cb & a \\ e_4 & e_5 & v & u \end{pmatrix}$ and the transpositions $\tau_1 = \begin{pmatrix} u & e_4 & e_5 & v \\ a^{-1} & a & v & v \\ e_4 & u & e_5 & v \end{pmatrix}, \tau_2 = \begin{pmatrix} v & e_5 & e_4 & u \\ cb & b^{-1}c^{-1} & v & u \\ e_5 & v & e_4 & u \end{pmatrix}$ and $\tau_3 = \begin{pmatrix} u & v & w \\ a^{-1}cb & b^{-1}c^{-1}a & w \\ v & u & w \end{pmatrix}$.
- A computation shows that

$$\tau_3\tau_2\tau_1\tau = \begin{pmatrix} e_4 & e_5 & v & u \\ v & v & v & (a^{-1}cba)^2 \\ e_4 & e_5 & v & u \end{pmatrix},$$

which is not a product of transpositions, since $G_u^u \cong \mathbb{Z}$. In particular, τ is not in $\ker(I)$.

THANK YOU!

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Homology of $\mathcal{G}(G,E)$

- Given a self-similar action (G,E), there is a cocycle
$$\rho:\mathcal{G}(G,E)\to\mathbb{Z},\quad \rho([\alpha,g,\beta;\xi])=|\alpha|-|\beta|$$
with kernel
$$\mathcal{N}(G,E)=\bigcup_{k\geq 1}\mathcal{N}_k(G,E),\ \text{where}$$
$$\mathcal{N}_k(G,E)=\{[\alpha,g,\beta;\xi]\in\mathcal{G}(G,E):|\alpha|=|\beta|=k\}\cong(E^\infty\rtimes G)\times R_k,$$
and R_k is an equivalence relation on E^k.
- There is a homomorphism $\tau_k:\mathcal{N}_k(G,E)\to G$, $[\alpha,g,\beta;\xi]\mapsto g$ with kernel $E^\infty\times R_k$.
- Since $\mathcal{N}(G,E)$ is equivalent to the skew product $\mathcal{G}(G,E)\times_\rho\mathbb{Z}$, we have an exact sequence
$$0\longleftarrow H_0(\mathcal{G}(G,E))\longleftarrow H_0(\mathcal{N}(G,E))\overset{id-\rho_*}{\longleftarrow}H_0(\mathcal{N}(G,E))\longleftarrow H_1(\mathcal{G}(G,E))$$
$$\uparrow$$
$$\cdots\longrightarrow H_2(\mathcal{N}(G,E))\longrightarrow H_2(\mathcal{G}(G,E))\longrightarrow H_1(\mathcal{N}(G,E))\overset{id-\rho_*}{\longrightarrow}H_1(\mathcal{N}(G,E))$$
where ρ_* is the map induced by the action $\hat{\rho}:\mathbb{Z}\curvearrowright\mathcal{G}(G,E)\times_\rho\mathbb{Z}$ which takes (γ,n) into $(\gamma,n+1)$.
- This allows to compute $H_*(\mathcal{G}(G,E))$ in some cases and to compare it with $K_*(C^*(G,E))$.