Noncommutative Choquet theory and noncommutative majorization

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Theorem (Kadison 1951)

A function system F is unitally order isomorphic to the function system $\mathrm{A}(C)$ of continuous affine functions on its state space $C = S(F)$

$$F \to \mathrm{A}(C) : f \to \hat{f} \quad \textit{where} \quad \hat{f}(x) = x(f) \quad \textit{for} \quad x \in C.$$

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The convex structure of C reveals itself in the interplay between the function system $\mathrm{A}(C)$ and the convex functions in $C(C)$: if $f \in \mathrm{C}(C)$ is convex, then $f(x) = \sup\{a(x) : a \in \mathrm{A}(C),\ a \leq f\}$.

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The **Choquet order** on $\mathrm{Prob}(C)$ is the partial order defined by

$$\mu \prec \nu \quad \text{if} \quad \mu(f) \leq \nu(f) \quad \text{for all convex } f \in \mathrm{C}(C).$$

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For $x \in C$, the set $\{\mu \in \mathrm{Prob}(C) : \mu|_{\mathrm{A}(C)} = x\}$ has a unique minimal element δ_x. By Zorn, it always contains at least one maximal element.

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Note: In finite dimensions, this is Carathéodory's theorem.

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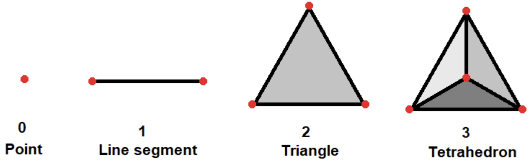

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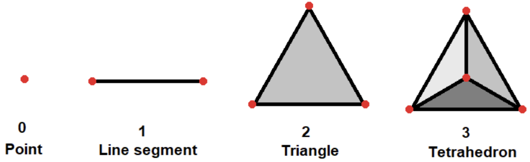

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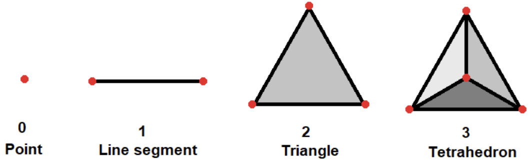

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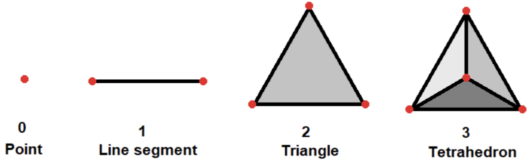

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4. Invariant probability measures $\mathrm{Prob}(X)^G$ on a compact G-space X.

Two applications of simplices

Theorem (Namioka-Phelps 1969)

A compact convex set C is a simplex iff $\mathrm{A}(C)$ is nuclear.

is evident that $S^\wedge \subset S$, and the following theorem gives conditions under which $S^\wedge = S$. The validity of "(c) implies (a)" was suggested to us by E. Effros.

THEOREM 1.4. *Let S_1 be the state space of (E_1, P_1, u_1); then the following assertions are equivalent*

(a) *S_1 is a simplex.*

(b) *S_1 is simplex-like.*

(c) *For any partially ordered linear space with order unit (E_2, P_2, u_2), the two state spaces resulting from the two orderings on $E_1 \otimes E_2$ coincide.*

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Theorem (Glasner-Weiss 1997)

A locally compact group G has property (T) iff $\mathrm{Prob}(X)^G$ is a Bauer simplex for every compact G-space X.

Some history of noncommutative convexity

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Note: missing numerous developments in operator spaces/systems (e.g. Choi-Effros, Effros-Ruan), matrix convexity and real algebraic geometry.

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Geometric side
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Missing a theory of nc continuous functions, convex nc functions, nc measures, etc. Analogy with Taylor-Voiculescu theory of nc analytic functions.

Classical notions are essential to fully develop the classical theory. In my view, operator space theory is incomplete without these.

Noncommutative convexity

Definition (DK2019)

A **compact nc convex set** over a dual operator space E is a graded set $K = \coprod_{n \leq \kappa} K_n$ with $K_n \subseteq M_n(E)$ such that each K_n is compact in the dual topology on $M_n(E)$ and K is closed under nc convex combinations:

$$\sum \alpha_i^* x_i \alpha_i \in K_n$$

for $x_i \in K_{n_i}$ and $\alpha_i \in M_{n,n_i}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

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Key example: Let S be an operator system. The **nc state space** of S is

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Theorem (Arveson 2007, DK 2015)

A compact nc convex set is the closed convex hull of its extreme points.

The extreme boundary ∂K can be identified with an (often very complicated) subset of the irreducible representations of $\mathrm{C}^*(\mathrm{A}(K))$. So necessary to allow $n = \infty$.

Noncommutative functions

Definition (DK2019)

Let K be a compact nc convex set. A function $f : K \to \coprod M_n$ is an **nc function** if it is graded, respects direct sums and is equivariant with respect to unitaries:

1. $f(K_n) \subseteq M_n$ for all n
2. $f(\oplus x_i) = \oplus f(x_i)$ for all $x_i \in K_{n_i}$
3. $f(\alpha^* x \alpha) = \alpha^* f(x) \alpha$ for all $x \in K_n$ and unitaries $\alpha \in M_n$

The function f is **affine** if in addition it is equivariant with respect to isometries:

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Note: Nonzero nc functions can be zero on K_n for $n < \infty$. Similarly, discontinuous nc functions can be continuous on K_n for $n < \infty$. More justification for $n = \infty$.

Noncommutative functions

Theorem (Webster-Winkler 1999, DK 2019)

An operator system S is unitally completely order isomorphic to the operator system $\mathrm{A}(K)$ of continuous nc affine functions on its nc state space $K = S_{nc}(S)$.

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Can be viewed as very special case of the noncommutative Stone-Weierstrass problem. Proof uses noncommutative Gelfand representation theorem of Takesaki (1967) and Bichteler (1969).

Example: The Cuntz operator system

Let $S = \mathrm{span}\{1, v_1, v_1^*, \ldots, v_d, v_d^*\}$ be the operator system spanned by the canonical generators of the Cuntz algebra $\mathcal{O}_d$, i.e. $v_i^* v_j = \delta_{ij}1$. Then $S \cong \mathrm{A}(K)$, where $K = \sqcup_{n \leq \aleph_0}$ is the nc d-ball

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The continuous nc function on K are uniform limits of *-polynomials in d-noncommuting variables, e.g. for $d = 2$,

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Let $S = \text{span}\{1, v_1, v_1^*, \ldots, v_d, v_d^*\}$ be the operator system spanned by the canonical generators of the Cuntz algebra $\mathcal{O}_d$, i.e. $v_i^* v_j = \delta_{ij} 1$. Then $S \cong \mathrm{A}(K)$, where $K = \sqcup_{n \leq \aleph_0}$ is the nc d-ball

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Note: the operator system spanned by freely independent semicirculars provides a (very interesting) example with ∂K a complicated subset of the irreducible representations of the C*-algebra it generates.

NC convex functions

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A self-adjoint-valued nc function $f \in C(K)$ is **convex** if its epigraph

$$\operatorname{Epi}(f) = \coprod_n \{(x, \alpha) : f(x) \leq \alpha\} \subseteq \coprod_n K_n \times M_n$$

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Then K is a compact nc convex set with $K_1 = I$. A self-adjoint function $f \in C(K)$ is convex as an nc function iff the restriction $f|_{K_1}$ is operator convex, i.e.

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta)$$

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Essentially the Hansen-Pedersen-Jensen inequality for operator convex functions.

Noncommutative Choquet order

Let K be a compact nc convex set. An nc state $\mu : C(K) \to M_n$ **represents** $x \in K$ or has barycenter x if $\mu|_{\mathrm{A}(K)} = \delta_x$.

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Theorem (NC Choquet-Bishop-de Leeuw - DK 2019)

An nc state $\mu \in S_{nc}(C(K))$ is maximal iff it is supported on the extreme boundary ∂K in a certain precise sense. Hence every point in K has a representing nc state supported on ∂K.

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A compact nc convex set K is an **nc simplex** if each point $x \in K$ has a unique maximal representing nc state.

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Theorem (KS 2021, K-Kim-Manor 2021)

A locally compact group G has property (T) iff $S_{nc}(A)^G$ is a Bauer simplex for every G-C-algebra A*

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Definition (Hardy-Littlewood-Pólya 1929)

For $a, b \in \mathbb{R}^n$, a is **majorized** by b, written $a \prec b$ if

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Choquet-theoretic perspective: Let $A = \mathrm{Diag}(a)$, $B = \mathrm{Diag}(b)$ and let $C \subseteq \mathbb{R}$ be a closed interval containing their spectrum. Define $\mu_A, \mu_B \in \mathrm{Prob}(C)$ by

$$\mu_A(f) = \mathrm{Tr}(f(A)),\ \mu_B(f) = \mathrm{Tr}(f(B)).$$

Then $a \prec b$ iff $\mu_A \prec_c \mu_B$.

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Let (M, τ) be a finite von Neumann algebra.

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Note: (4) utilizes the Birkhoff-von Neumann theorem.

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Noncommutative majorization

Let (M, τ) be a finite von Neumann algebra.

Definition (K-Marcoux-Skoufranis 2023)

For tuples $a = (a_i), b = (b_i)$ in M, a is **majorized** by b, written $a \prec b$ if

$$f(a) \leq f(b) \quad \text{for all convex nc functions } f \in C(K),$$

where K is a sufficiently large compact nc convex set.

Note: Elements are not required to be self-adjoint.

Recall: a map $\phi : M \to M$ is **doubly stochastic** if it is normal unital completely positive and trace preserving.

Theorem (KMS 2023)

For tuples $a = (a_i), b = (b_i)$ in M_{sa}, the following are equivalent:

1. $a \prec b$,
2. $a_i = \phi(b_i)$ *for all* i *for a doubly stochastic map* $\phi : M \to M$.

Note: Not true in general that a belongs to the unitary orbit of b (i.e. that ϕ can be chosen to be mixed unitary) even for $|a| = |b| = 1$ in the non-self-adjoint case. Counterexamples utilize negative solution to the asymptotic Birkhoff-von Neumann conjecture (Haagerup-Musat 2011).

Key idea

Key idea is equivalence between nc Choquet order and the "dilation order:" for nc states μ, ν on $C(K)$, $\mu \prec_{nc} \nu$ iff there is a Stinespring representations (π_μ, v) of μ such that $\pi_\mu|_{\mathrm{A}(K)}$ dilates to a Stinespring representation (π_ν, v) of ν, i.e.

$$\pi_\nu|_{\mathrm{A}(K)} = \begin{bmatrix} \pi_\mu|_{\mathrm{A}(K)} & * \\ * & * \end{bmatrix}.$$

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More generally, can characterize existence of ucp maps between tuples in M that preserve arbitrary ucp maps on M, e.g. states, conditional expectations.

Thanks!