Basis Problems and Expansions, III

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What is a basis problem?

Definition

Given a quasi-ordered class $(\mathcal{K}, \leq)$ of mathematical structures of the same type, we say that $\mathcal{K}_0 \subseteq \mathcal{K}$ is a **basis** *of $\mathcal{K}$ if for every $K \in \mathcal{K}$ there is $K_0 \in \mathcal{K}_0$ such that $K_0 \leq K$.*

Problem

Suppose $\mathcal{K}_0$ is a downwards closed subclass of a given quasi-ordered class $(\mathcal{K}, \leq)$. Can one characterize $\mathcal{K}_0$ by **forbidding finitely many** *members of $\mathcal{K}$?*

Basis problem for gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$

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Notation:
Fix a countable index set N. For $a, b \subseteq N$, set

$$a \subseteq^* b \text{ iff } a \setminus b \text{ is finite},$$

$$a \perp b \text{ iff } a \cap b \text{ is finite}.$$

For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(N)$, set

$$\mathcal{A} \perp \mathcal{B} \text{ iff } (\forall a \in \mathcal{A})\,(\forall b \in \mathcal{B})\ \ a \perp b.$$

$$\mathcal{A}^{\perp} = \{b : (\forall a \in \mathcal{A})\ b \cap a \text{ is finite}\}.$$

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A **preideal** on a countable set N is a family I of subsets of N such that if $x \in I$ and $y \subseteq x$ is infinite, then $y \in I$.

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1. We say that Γ is **separated** if there exist $a_0, \ldots, a_{n-1} \subseteq N$ such that $\bigcap_{i \in n} a_i = \emptyset$ and $x \subseteq^* a_i$ for all $x \in \Gamma_i$, $i \in n$.

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2. We say that Γ is an $\mathfrak{X}$-**gap** if it is **not separated**, but $\bigcap_{i \in A} x_i =^* \emptyset$ whenever $x_i \in \Gamma_i$, $A \in \mathfrak{X}$.

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When $\mathfrak{X} = [n]^2$ an $\mathfrak{X}$-gap will be called an n-**gap**.

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Hausdorff gaps

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For $k > 2$, *assuming Martin's axiom, there are no* $< \mathfrak{c}$*-generated* k*-gaps in* $\mathcal{P}(\omega)/\mathrm{FIN}$.

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A 2-gap $(\mathcal{A}, \mathcal{B})$ for which $\mathcal{A}$ and $\mathcal{B}$ are σ-directed under the inclusion modulo finite is called a **Hausdorff gap***.*

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Theorem (T. 1996)

There are **no analytic** *Hausdorff gaps.*

Some applications of analytic gaps

Theorem (T., 1999)

The class of **non-metrizable separable compact spaces** *of Baire-class-1 functions defined on a Polish space has the* **3-element basis** $\{S, D, P\}$, *where* S *is the split-interval,* D *the (separable version of the) Alexandrov duplicate of the Cantor set, and* P *the one-point compactification of the Cantor tree space.*

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Theorem (Argyros-Dodos-Kanellopoulos, 2008)

Every infinite-dimensional **dual** *Banach space has an infinite-dimensional quotient with a Schauder basis.*

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For Γ and Δ two n_*-gaps on countable sets N and M, respectively, we say that

$$\Gamma \leq \Delta$$

if there exists a one-to-one map $\phi : N \to M$ such that for $i < n$,

1. if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$.
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Two n_*-gaps Γ and Γ' are called **equivalent** if $\Gamma \leq \Gamma'$ and if $\Gamma' \leq \Gamma$.

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- *When* Γ *is a n-gap, the second condition can be substituted by saying that if* $x \in \Gamma^\perp$ *then* $\phi(x) \in \Delta^\perp$.

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- *Another observation is that the above definition implies that* $\phi(x) \in \Delta_i^{\perp\perp}$ *if and only if* $x \in \Gamma_i^{\perp\perp}$, *and* $\phi(x) \in \Delta^{\perp}$ *if and only if* $x \in \Gamma^{\perp}$.

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- *There are other variants of the order* $\leq$ *between gaps but all lead essentially to the same theory.*

Definition

An analytic n_*-gap Γ is said to be a **minimal analytic** n_***-gap** if for every other analytic n_*-gap Δ, if $\Delta \leq \Gamma$, then $\Gamma \leq \Delta$.

Finite Basis Theorem for Gaps

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Fix a positive integers n.

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Up to permutations there is exactly 5 minimal analytic 2-gaps. Most of them already show up in the literature.

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Corollary

Let $\mathcal{U}$ be a countable family of pairwise disjoint analytic open subsets of $\beta\mathbb{N} \setminus \mathbb{N}$, and let $\{U_0, U_1, U_2\} \subseteq \mathcal{U}$ be pairwise distinct such that $\overline{U_0} \cap \overline{U_1} \cap \overline{U_2} \neq \emptyset$.
Then, there exists a point $x \in \overline{U_0} \cap \overline{U_1} \cap \overline{U_2}$ such that

$$|\{U \in \mathcal{U} : x \in \overline{U}\}| \leq 61.$$

Moreover, 61 *is optimal in this result.*

For a positive integer n, let $J(n)$ be the minimal for which the conclusion of this Corollary is true with 3 replaced with n. Then we have the following:

n	1	2	3	4	5	6	7	$\cdots$
$J(n)$	1	8	61	480	3881	31976	266981	$\cdots$

The expansion

Definition

Fix an integer $n \geq 2$ and consider $n^{<\omega}$ as a tree ordered by end-extension $\leq$. For $i < n$, a **chain of type** $[i]$ *is a chain C of the tree with the property that if $s < t$ are two consequtive nodes of C then*

- $s^\frown i \leq t$,
- $t(k) \leq i$ *for all* $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

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Theorem (Aviles-T., 2014)

If $\Gamma = \{\Gamma_i : i \in n\}$ are analytic preideals on a countable index set N which are not separated, then there exists a permutation $\varepsilon : n \to n$ and a one-to-one map $u : n^{<\omega} \to N$ such that $u(x) \in \Gamma_{\varepsilon(i)}$ whenever x is a chain of type $[i]$, $i < n$.

A partition theorem for trees

For a fixed natural number k, we denote by W_k the set of all finite sequences of natural numbers from $\{0, \ldots, k\}$ that start by k, that is

$$W_k = \{ \ (t_0, t_1, \ldots, t_p) \ : \ t_0 = k, \ t_i \in \{0, \ldots, k\}, \ i = 1, \ldots, p \ \}$$

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Definition

We will say that a subset $F \subset m^{<\omega}$ is **closed** if it satisfies:

1. If $s, t \in F$, then $s \wedge t \in F$
2. If $s = t^\frown r_1^\frown \cdots ^\frown r_k$ with $t, s \in F$, $r_1 \in W_{i_1}, \dots, r_k \in W_{i_k}$, $i_1 < i_2 < \cdots < i_k$, then $t^\frown r_1 \in F$ (therefore also $t^\frown r_1^\frown r_2 \in F$, etc.)

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Given $F \subset m^{<\omega}$, let $\langle F \rangle$ be the closed set **generated by** F.

Definition

Consider sets $X \subset m^{<\omega}$, $Y \subset n^{<\omega}$. A function $f : X \to Y$ is called an **equivalence** if it is the restriction of a bijection $g : \langle X \rangle \to \langle Y \rangle$ satisfying the following for all $s, t \in \langle X \rangle$

1. $g(t \wedge s) = g(t) \wedge g(s)$,
2. $g(t) \prec g(s)$ iff $t \prec s$,
3. If $t \leq s$ then for every k, $s \setminus t \in W_k$ iff $g(s) \setminus g(t) \in W_k$

where $\prec$ is the notation of the lexicographical orderings on $m^{<\omega}$ and $n^{<\omega}$.

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Definition

A sequence $\{w_0, w_1, \ldots\} \subset W_k$ is called **basic** if

$$|w_i| > \sum_{j<i} |w_j|$$

for every i.

Define $T : W_k \to W_{k-1}$ by

$$T(w)(i) = \max\{0, w(i) - 1\}.$$

Let
$T^{(0)} : W_k \to W_k$ be the identity map,
$T^{(1)} = T$,
$T^{(j)} : W_k \to W_{k-j}$ be the j-th iterate of T.
Let $T_i = T^{(k-i)} : W_k \to W_i$ using the same subindex for T_i as for the range space W_i.

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Definition

Let $m \leq n < \omega$. A function

$$\psi : m^{<\omega} \to n^{<\omega}$$

will be called a **basic embedding** if there exists a basic sequence $\{w_s : s \in m^{<\omega}\} \subset W_{m-1}$ such that for every $t \in m^{<\omega}$ and for every $i \in m$, we have that

$$\psi(t^\frown i) = \psi(t)^\frown T_i(w_{t^\frown i}).$$

Proposition

If $\psi : m^{<\omega} \to n^{<\omega}$ *is a basic embedding, then* X *is equivalent to* $\psi(X)$ *for every set* $X \subset m^{<\omega}$.

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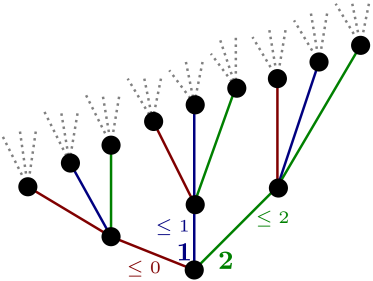
Definition

A **basic subtree** of $m^{<\omega}$ is the range of a basic embedding $\psi : m^{<\omega} \to m^{<\omega}$.

For a fixed set $X_0 \subset m^{<\omega}$, let us say that Y is an X_0-**set** if Y is equivalent to X_0.

Example

A basic subtree of $3^{<\omega}$*:*



Theorem (Partition Theorem for Trees)

Fix a set $X_0 \subset m^{<\omega}$. Then for every finite partition of the X_0-subsets of $m^{<\omega}$ into Souslin-measurable subsets, there exists a basic subtree $T \subset m^{<\omega}$ all of whose X_0-subsets lie in the same piece of the partition.

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Theorem (Finite Partition Theorem for Trees)

For every positive integers m,
every finite $X_0 \subseteq m^{<\omega}$,
and every positive integer k
there is a positive integer l such that
for every 2-coloring of the family of all X_0-subsets of $m^{<l}$
there is $Y \subseteq m^{<l}$ equivalent to $m^{<k}$ such that
the family of all X_0-subsets of Y is monochromatic.

Expansion problem for $\mathbb{Q}$

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Theorem (Laver 1970)

For every natural number k and every set $R \subseteq \mathbb{Q}^k$ there is $M \subseteq \mathbb{Q}$ **order-isomorphic to** $\mathbb{Q}$ *such that $R \cap M^k$ is $(\mathbb{Q}, \leqslant, <')$*-**canonical relation** *on M, where $\leqslant$ is the usual ordering on $\mathbb{Q}$ and where $<'$ is a well-order of $\mathbb{Q}$ of order-type ω.*

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Theorem (Laver 1970, Devlin 1979)

For every positive integer k there is an relations D_k on $\mathbb{Q}^{[k]}$ that has exactly $t_k = \tan^{2k-1}(0)$ classes such that for every equivalence relation E on $\mathbb{Q}^{[k]}$ with finitely many classes there is $M \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that the restriction of E on $M^{[k]}$ is coarser that the restriction of D_k on $M^{[k]}$.

Other homogeneous structures

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- *Under which condition on $\mathbb{A}$ we can find an expansion $\mathbb{A}^*$ with finitely many relations such that every subset R of some finite power $\mathbb{A}^k$ is $\mathbb{A}^*$-**canonical** when restricted to some substructure of $\mathbb{A}$ isomorphic to $\mathbb{A}$?*

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- *Under which additional assumptions (if any) can we conclude that on any finite symmetric power $\mathbb{A}^{[k]}$ there is* **the finest** *canonical equivalence relation with finitely many classes?*

The expansion problem for $\mathrm{top}(\mathbb{Q})$

We now consider $\mathbb{Q}$ as a topological space.

Theorem (Baumgartner, 1986)

There is a mapping $f : \mathbb{Q}^{[2]} \to \mathbb{N}$ *such that* $f''X^{[2]} = \mathbb{N}$ *for every* $X \subseteq \mathbb{Q}$ **homeomorphic** *to* $\mathbb{Q}$.

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Theorem (T., 1994)

There is an equivalence relation E_{osc} *on* $\mathbb{Q}^{[2]}$ *with infinitely many classes* $e_1, e_2, \ldots, e_k, \ldots$ *such that if for some positive integer* k *the closure* $\overline{X}$ *of some subset* X *of* $\mathbb{Q}$ *has its* k*th Cantor-Bendixson derivative nonempty then*

$$X^{[2]} \cap e_i \neq \emptyset \text{ for all } 2 \leq i \leq 2k.$$

Moreover, if X *is not a discrete subspace of* $\mathbb{Q}$ *then* $X^{[2]} \cap e_1 \neq \emptyset$.

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For every positive integer p and every continuous mapping

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there is $X \subseteq \mathbb{Q}$ homeomorphic to $\mathbb{Q}$ such that f is constant on $X^{[2]}$.

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Theorem (T., 1994)

The class of equivalence relations on $\mathbb{Q}^{[2]}$ with **open equivalence classes** *has 26-element Ramsey basis.*

Basis problems for $\mathbb{R}$

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Theorem (Sierpinski, 1933)

Let $\leqslant$ be the usual lexicographic ordering of $2^{\mathbb{N}}$, let $<'$ be a well-ordering of $2^{\mathbb{N}}$ and let $\mathbb{S}$ denote the expanded structure $(2^{\mathbb{N}}, \Delta, \leqslant, <')$. Then for every positive integer k the finest $\mathbb{S}$-canonical equivalence relation $\sim^k_{\mathbb{S}}$ on $(2^{\mathbb{N}})^{[k]}$ that has $k!(k-1)!$ many classes has the property that every $X \subseteq 2^{\mathbb{N}}$ **homeomorphic to** $\mathbb{Q}$ *realizes all the classes.*

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Conjecture (Galvin, 1970)

For every positive integer k every equivalence relation on $\mathbb{R}^{[k]}$ with finitely many classes is $\mathbb{S}$-canonical when restricted to some **uncountable set** *$X \subseteq \mathbb{R}$.*

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Theorem (Shelah, 2000)

For $k = 2$ the Galvin conjecture is **consistent** *with rather large continuum.*

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For every positive integer k there is

$$f : [\omega_k]^{k+1} \to \omega$$

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Galvin's Conjecture implies $2^{\aleph_0} > \aleph_\omega$.

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Galvin's Conjecture implies $2^{\aleph_0} > \aleph_\omega$.

Question

Is Galvin's Conjecture consistent with $2^{\aleph_0} = \aleph_{\omega+1}$?

Borel version of the expansion problem for $\mathbb{R}$

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The Cantor space $2^{\mathbb{N}}$ as the Borel structure:

$$(2^{\mathbb{N}}, \leqslant, \Delta)$$

where $\leqslant$ is the lexicographical ordering and Δ the distance function:

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

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Theorem (Galvin, 1968, Blass 1981)

- *For every positive integer k every Borel set $R \subseteq (2^{\mathbb{N}})^k$ is $(2^{\mathbb{N}}, \leqslant, \Delta)$-**canonical** on some perfect set $P \subseteq 2^{\mathbb{N}}$.*
- *Among $(2^{\mathbb{N}}, \leqslant, \Delta)$-canonical Borel equivalence relation on a given finite symmetric power $(2^{\mathbb{N}})^{[k]}$ with **finitely many classes** there is the finest one which has exactly $(k-1)!$ many classes.*

Theorem (Taylor 1979, Lefmann 1983, Vuksanovic 2008, Vlitas 2014)

- *There is exactly* **two** $(2^{\mathbb{N}}, \leqslant, \Delta)$*-canonical Borel equivalence relations on* $(2^{\mathbb{N}})^{[2]}$ *with* **countably many classes**: $\top$ *and* E_{Δ}.

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- *There is exactly* **seven** $(2^{\mathbb{N}}, \leqslant, \Delta)$*-canonical Borel equivalence relations on* $(2^{\mathbb{N}})^{[2]}$ *given by the following* **seven conditions** *on given two pairs* $x_0 < x_1$ *and* $y_0 < y_1$:
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  3. $x_1 = y_1$,
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For two structures $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same type, set

$$\binom{\boldsymbol{B}}{\boldsymbol{A}} = \{\boldsymbol{A}' : \boldsymbol{A}' \text{ is a substructure of } \boldsymbol{B} \text{ isomorphic to } \boldsymbol{A}\}.$$

For $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$ of the same type and cardinals λ and τ, let

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,\tau}$$

denote the statement that for every coloring $\chi : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \lambda$ there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ such that χ on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ has $\leq \tau$ values.

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$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda} \quad \text{iff} \quad \boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,1},$$

$$\boldsymbol{C} \to [\boldsymbol{B}]^{\boldsymbol{A}}_{\lambda} \quad \text{iff} \quad \boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,\lambda-1}.$$

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Fix a positive integer k *and let* $t_k = \tan^{(2k-1)}(0)$ *and consider the rationals* $\mathbb{Q}$ *as ordered set.*

- $\mathbb{Q} \to (\mathbb{Q})^k_{l,t_k}$ *for all* $l < \omega$.
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Theorem (Sierpinski 1933, Raghavan-T. 2020)

- $\mathbb{R} \not\to (\mathrm{top}(\mathbb{Q}))^2_2$.
- $\mathbb{R} \to (\mathrm{top}(\mathbb{Q}))^2_{l,2}$ *for all* $l < \omega$.

The 3-dimensional expansion problem for $\mathrm{top}(\mathbb{Q})$

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For every $k < \omega$ and every Hausdorff space X of cardinality $\aleph_k$,

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If for some integer $k \geq 1$,

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Corollary (Raghavan - T., 2022)

If $\mathbb{R} \to (\mathrm{top}(\mathbb{Q}))^{3}_{13,12}$ then $|\mathbb{R}| \geq \aleph_2$.

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For $\boldsymbol{A} \in \mathcal{K}$, let $t(\boldsymbol{A}, \mathcal{K})$ be the **minimal number** *t (if it exists) such that for every $\boldsymbol{B}$ in $\mathcal{K}$ and $l < \omega$ there exists $\boldsymbol{C}$ in $\mathcal{K}$ such that*

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- *Finite linearly ordered sets have Ramsey degree 1 in the class of all* **finite linear orderings***, i.e., $\mathbb{Q} \to (n)^k_l$ for all $k, l, n < \omega$.*

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- *Complete graphs have Ramsey degree 1 in the class of all* **finite graphs**, *i.e., $\mathcal{R} \rightarrow (\mathbf{G})^{K_k}_l$ for all finite graphs $\mathbf{G}$ and $k, l < \omega$.*

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For $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$, set

$$X_{\boldsymbol{F}^*}^{\boldsymbol{A}} = \{(R_i^* : i < n) \in \prod_{i<n} 2^{\boldsymbol{A}^{k_i}} : (\boldsymbol{A}, R_0^*, ..., R_{n-1}^*) \in \mathrm{Age}(\boldsymbol{F}^*)\}.$$

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Proposition

If $\mathrm{Age}(\boldsymbol{F}^*)$ *has the* **Ramsey property** , *then*

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and so, in particular $t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) < \infty$ *for all* $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$.

Definition

For $\boldsymbol{F}$ *and* $\boldsymbol{F}^*$ *as above, we say that* $\boldsymbol{F}^*$ *has the* **expansion property** *relative to* $\boldsymbol{F}$ *whenever*

$$\forall \boldsymbol{A}^* \in \mathrm{Age}(\boldsymbol{F}^*) \;\; \exists \boldsymbol{B} \in \mathrm{Age}(\boldsymbol{F}) \;\; \forall \boldsymbol{B}^* \in \mathrm{Age}(\boldsymbol{F}^*)$$

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Proposition

If the expansion $\boldsymbol{F}^*$ has both the **Ramsey property** *and the* **expansion property** *relative to $\boldsymbol{F}$, then*

$$t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) = t_{\boldsymbol{F}^*}(\boldsymbol{A}) \text{ for all } \boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F}).$$

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Definition

An expansion $\boldsymbol{F}^* = (\boldsymbol{F}, (R_i : i < \omega))$ *of* $\boldsymbol{F}$ *is* **precompact** *if its* **age** *restricted to every finite substructure of* $\boldsymbol{F}$ *is* **finite***.*

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The point of this definition (among other things) is that the corresponding version of $t_{\boldsymbol{F}^*}(\boldsymbol{A})$ *is still finite.*

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Proposition

The following are equivalent for a given countably infinite homogeneous structure $\boldsymbol{F}$*:*

- $t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) < \infty$ *for every* $\boldsymbol{A} \in \boldsymbol{F}$.
- *There is a* **Ramsey precompact expansion** $\boldsymbol{F}^*$ *of* $\boldsymbol{F}$.

Proposition

If there is a **Ramsey precompact expansion** *of* $\boldsymbol{F}$ *then there is one that also has the* **expansion property**.

Topological dynamics reformulation

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Theorem (Zucker 2014)

Let $\boldsymbol{F}$ *be a countable locally finite ultrahomogeneous structure. If the group* $\mathrm{Aut}(\boldsymbol{F})$ *has* **metrizable universal minimal flow** *then* $t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) < \infty$ *for all* $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$.

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