Basis Problems and Expansions, II

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What is a basis problem?

Definition

Given a quasi-ordered class $(\mathcal{K}, \leq)$ *of mathematical structures of the same type, we say that* $\mathcal{K}_0 \subseteq \mathcal{K}$ *is a* **basis** *of* $\mathcal{K}$ *if for every* $K \in \mathcal{K}$ *there is* $K_0 \in \mathcal{K}_0$ *such that* $K_0 \leq K$*.*

Versions of basis problems

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Problem

Suppose $\mathcal{K}_0$ is a downwards closed subclass of a given quasi-ordered class $(\mathcal{K}, \leq)$. Can one characterize $\mathcal{K}_0$ by **forbidding finitely many** *members of $\mathcal{K}$?*

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- *Can one characterize in this way the class of* **countable** *linear orderings in the class of all linear orderings?*
- *Can one characterize in this way the class of* **metrizable** *compact spaces in the class of all compact spaces?*

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The class of **infinite** *linear orderings has basis* $\{\omega^*, \omega\}$.

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If there a finite list $\{L_1, L_2, ..., L_n\}$ *of* **uncountable linear orderings** *such that* $\{L_1, L_2, ..., L_n\}^\perp$ *is the class of* **countable linear orderings**.

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A quasi-ordered set $(Q, \leq)$ *is* **well-quasi-ordered**, *wqo, if for every sequence* $(q_i : i < \omega)$ *of elements of* Q *there exist* $i < j$ *such that* $q_i \leq q_j$.

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Corollary (Laver, 1971)

Every *class of* σ-**scattered** *linear orderings has a* **finite basis**.

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- For every positive integer k, the set $[\mathbb{N}]^k = \{F \subseteq \mathbb{N} : |F| = k\}$ is a barrier.
- The family $\mathcal{S} = \{F \subseteq \mathbb{N} : |F| = \min(F) + 1\}$ is a barrier of infinite rank.

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Theorem (Nash-Williams, 1965)

For every barrier $\mathcal{F}$ on $\mathbb{N}$ and every positive integer p, every

$$f : \mathcal{F} \to \{0, 1, ...p - 1\}$$

is constant on $\mathcal{F} \restriction M$ for some infinite $M \subseteq \mathbb{N}$.

Definition (Nash-Williams 1965)

A quasi-ordered set $(Q, \leq)$ is **better quasi-ordered** *if for every barrier $\mathcal{F}$ on $\mathbb{N}$ and*

$$f : \mathcal{F} \to Q$$

there exist $s, t \in \mathcal{F}$ such that $s \triangleleft t$ and $f(s) \leq f(t)$.

Notation: $s \triangleleft t$ means that s is an initial segment of $u = s \cup t$ and that $t = u \setminus \{\min(u)\}$.

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Theorem (Simpson 1985)

A quasi-ordered set $(Q, \leq)$ *is* **better quasi-ordered** *iff for every* **Borel map**

$$f : \mathbb{N}^{[\infty]} \to Q$$

there exists $M \in \mathbb{N}^{[\infty]}$ *such that* $f(M) \leq f(M \setminus \{\min(M)\})$.

For a given quasi-ordered set Q let $Q^{\mathrm{Ord}} = \bigcup_{\alpha \in \mathrm{Ord}} Q^\alpha$.

We quasi-order Q^{Ord} by letting

$$(x_\xi : \xi < \alpha) \leq_1 (y_\eta : \eta < \beta)$$

if there is a strictly increasing sequence $(\eta_\xi : \xi < \alpha)$ of ordinals $< \beta$ such that

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Theorem (Nash-Williams 1968)

If Q is bqo then so is Q^{Ord}.

Separable linear orderings

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Question

What is the minimal cardinal θ such that the class $\mathrm{Sep}(\theta)$ of separable linear ordering of cardinality $< \theta$ is well-quasi-ordered?

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This cardinal falls into the interval $[\aleph_1, 2^{\aleph_0}]$.

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Theorem (Baugartner 1973)

PFA implies that the class of separable linear orderings of cardinality smaller than continuum is well-quasi-ordered.

Definition

An **Aronszajn ordering** *is an uncountable linearly ordered set L such that*

- $\omega_1 \not\leq L$,
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- $S \not\leq L$ *for any uncountable separable linear ordering.*

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Theorem (Aronszajn-Kurepa 1935, Specker 1949)

There is an Aronszajn ordering.

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Theorem (Aronszajn-Kurepa 1935, Specker 1949)

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Theorem (Martinez-Ranero, 2011)

PFA implies that the class of **Aronszajn orderings** *is well-quasi-ordered.*

Corollary (Martinez-Ranero, 2011)

PFA implies that **every** *class of* **Aronszajn orderings** *has a finite basis.*

Theorem (Baumgartner, 1973)

PFA implies that the class of **uncountable separable orderings** *has a* **one-element basis**.

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Theorem (Moore, 2005)

PFA implies that the class of **Aronszajn orderings** *has basis* $\{C^*, C\}$, *where* C *is any uncountable linear ordering whose cartesian square is the union of countably many chains.*

Corollary (Baumgartner 1973, Moore 2005)

PFA implies that the class of **uncountable linear orderings** *has basis*

$$\{\omega_1^*, \omega_1, B, C^*, C\}$$

where B is any set of reals of cardinality $\aleph_1$ and where C is any uncountable linear ordering whose cartesian square is the union of countably many chains.

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Corollary (Baumgartner 1973, Moore 2005)

PFA implies that the class of **countable linear orderings** *is equal to $\{\omega_1^*, \omega_1, B, C^*, C\}^\perp$.*

Basis problems for trees

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Definition

For two trees S and T, by $S \leq_1 T$ we denote the fact that S can be **topologically embedded** *into T, i.e., that there is $f : S \to T$ which is* **strictly increasing** *and $\wedge$-preserving.*

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The class of **finite trees** *is well-quasi-ordered by the relation $\leq_1$.*

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*The class of σ-***scattered trees** *is well-quasi-ordered by the relation $\leq_1$.*

Corollary (Laver, 1978)

Every *class of σ-***scattered trees** *quasi-ordered by the relation $\leq_1$ has a finite basis.*

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Definition

For two trees S and T let $S \leq T$ if there is **strictly increasing** *map $f : S \to T$.*

Theorem (T. 2000)

The class $\mathcal{A}$ of **Aronszajn trees** *is not well-quasi-ordered even under the weaker relation $\leq$.*

Theory of Lipschitz trees

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Definition

A partial map g from a tree S into a tree T is **Lipschitz**, *if g is* **level preserving** *and*

$$\Delta(g(x), g(y)) \geq \Delta(x, y) \text{ for all } x, y \in \mathrm{dom}(g).$$

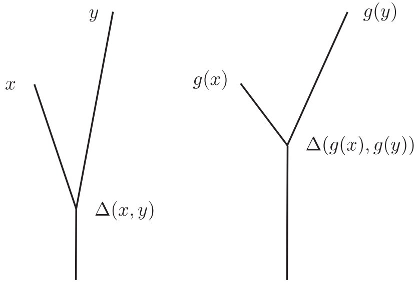

Figure: Lipschitz map on a tree

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Remark

Lipschitz trees do exist. For example any of the trees $T(\rho_0)$, $T(\rho_1)$, and $T(\rho_2)$ is Lipschitz, where ρ_0, ρ_1 and ρ_2 are the standards characteristics of walks on ω_1.

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Theorem (T. 2000)

Assuming PFA the class $(\mathcal{A}, \leq)$ of Aronszajn trees is generated by a discrete chain $\mathcal{L}$ of **Lipschitz trees** *such that $(\mathcal{L}/\mathbb{Z}, \leq)$ is the $\aleph_2$-saturated linear ordering of cardinality $\aleph_2 = 2^{\aleph_1}$.*

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Theorem (T. 2000)

Let S and T be Lipschitz trees.

- $\mathcal{U}(S)$ *and* $\mathcal{U}(T)$ *are* **uniform filters** *on* ω_1.
- *If the countable chain condition is productive* $\mathcal{U}(S)$ *and* $\mathcal{U}(T)$ *are* **uniform ultrafilters**.
- MA_{ω_1} *implies that* $S \equiv T$ *iff* $\mathcal{U}(S) = \mathcal{U}(T)$.

Notation: $S \equiv T$ *iff* $S \leq T$ *and* $T \leq S$.

The shift of a Lipschitz tree

Definition

Suppose g is a partial map from ω_1 into ω_1 and that T is a downward closed subtree of the tree

$$\{t : \alpha \to \omega : \alpha < \omega_1\}.$$

*Then the g-**shift** of T, denoted by $T^{(g)}$, is the downward closure of*

$$\{t^{(g)} : t \in T \restriction C_g\}$$

where $C_g = \{\delta < \omega_1 : g''\delta \subseteq \delta\}$ and where $t^{(g)}$ is defined by

$$t^{(g)}(\xi) = t(g(\xi))$$

if $\xi \in \mathrm{dom}(g)$; otherwise $t^{(g)}(\xi) = 0$.

When $g(\xi) = \xi + 1$ for all $\xi \in \mathrm{dom}(g)$, we denote $T^{(g)}$ by $T^{(1)}$.

Only one tree and only one ultrafilter

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Theorem (T., 2000)

- *If T is a Lipschitz tree and if g is a partial strictly increasing map from ω_1 to ω_1 such that $\mathrm{range}(g) \in \mathcal{U}(T)$, then the g-shift $T^{(g)}$ is also a Lipschitz tree.*

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- *Suppose T is a Lipschitz tree and g is a partial strictly increasing map from ω_1 to ω_1 such that $\mathrm{range}(g) \in \mathcal{U}(T)$. If g is* **regressive**, *then $T \not\leq T^{(g)}$.*

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  If g is **expanding** *then $T^{(g)} \not\leq T$.*

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- *Suppose T is a Lipschitz tree and g is a partial strictly increasing map from ω_1 to ω_1 such that $\mathrm{range}(g) \in \mathcal{U}(T)$. If g is* **regressive**, *then $T \not\leq T^{(g)}$. If g is* **expanding** *then $T^{(g)} \not\leq T$.*

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Corollary (T. 2000)

PFA implies that for every pair S and T of Lipschitz trees the ultrafilters $\mathcal{U}(S)$ and $\mathcal{U}(T)$ are Rudin-Keisler equivalent.

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Theorem (Moore 2005)

PFA implies that for every Aronszajn tree T *and every* $C \subseteq T$ *there is uncountable* $X \subseteq T$ *such that either*

$$\{s \wedge t : s, t \in X, s \neq t\} \subseteq C \text{ or } \{s \wedge t : s, t \in X, s \neq t\} \subseteq T \setminus C.$$

One selective ultrafilter

For a Lipschitz tree T fix a $\mathcal{U}(T)$-nowhere constant map $f : \omega_1 \to \omega$, and let

$$\mathcal{U}^f_\omega(T) = \{M \subseteq \omega : M \supseteq f''X \text{ for some } X \in \mathcal{U}(T)\}.$$

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$$\mathcal{U}^f_\omega(T) = \{M \subseteq \omega : M \supseteq f''X \text{ for some } X \in \mathcal{U}(T)\}.$$

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MA_{ω_1} *implies that* $\mathcal{U}^f_\omega(T)$ *is a* **selective ultrafilter** *on* ω *whose Rudin-Keisler class does not depend on* T *nor* f.

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Definition (Choquet 1968)

*An ultrafilter $\mathcal{V}$ on ω is **selective** (**Ramsey**) if it is non-principal and if for every $f : \omega \to \omega$ there is $M \in \mathcal{V}$ such that the restriction of f on M is either **constant** or **one-to-one**.*

Tukey reductions

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Definition

A partially ordered set P is **Tukey reducible** *to a partially ordered set Q, in notation $P \leq_T Q$, if there is a map $f : P \to Q$ that maps unbounded subsets of P to unbounded subsets of Q, or equivalently, a map $g : Q \to P$ which maps cofinal subsets of Q to cofinal subsets of P.*
When $P \leq_T Q$ and $Q \leq_T P$ we write $P \equiv_T Q$ and say that P and Q are **Tukey equivalent** *or* **cofinaly similar***.*

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Remark

In the class of (upwards) **directed** *posets $P \equiv_T Q$ is equivalent to saying that P and Q are* **isomorphic** *to cofinal subsets of a single directed poset R.*

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The directed set $[\theta]^{<\omega}$ *of finite subsets of some infinite cardinal* θ *realizes the* **maximal Tukey type** *among directed posets of cardinality at most* θ*.*

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Definition (W. Rudin, 1956)

An ultrafilter $\mathcal{U}$ on ω is a P-point if for every sequence $(A_n : n < \omega)$ of elements of $\mathcal{U}$ there is $B \in \mathcal{U}$ such that $B \setminus A_n$ is finite for all $n < \omega$.

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Remark

If $\mathcal{U}$ is a P-point ultraflter on ω then $\mathcal{U} \not\equiv_T \mathcal{U}_{\max}$.

Five cofinal types

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Theorem (T., 1985, 1996)

PFA implies that

$$1, \omega, \omega_1, \omega \times \omega_1 \text{ and } [\omega_1]^{<\omega}$$

are all Tukey types of **directed sets** *of cardinality at most* $\aleph_1$.

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Moreover, letting $D_0 = 1$, $D_1 = \omega$, $D_2 = \omega_1$, $D_3 = \omega \times \omega_1$, *and* $D_4 = [\omega_1]^{<\omega}$, *every* **partially ordered set** *of cardinality at most* $\aleph_1$ *is Tukey equivalent to one of these:*

- $\bigoplus_{i<5} n_i D_i$ $(i < 5, n_i < \omega)$,
- $\aleph_0 \cdot 1 \oplus \bigoplus_{i=2}^{4} n_i D_i$ $(2 \leq i < 5, n_i < \omega)$,
- $\aleph_0 \cdot \omega_1 \oplus n_4 [\omega_1]^{<\omega}$ $(n_4 < \omega)$,
- $\aleph_0 \cdot [\omega_1]^{<\omega}$,
- $\aleph_1 \cdot 1$.

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Remark

The topology of a basic order is uniquely determined by the order itself. It is the topology of sequential convergence where a sequence (x_n) is set to be convergent if $\limsup x_n = \liminf x_n$ and if all subsequences of (x_n) have further subsequences that are bounded.

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- *D is compact iff $D \equiv_T 1$.*
- *If D is analytic and not locally compact then $\mathbb{N}^{\mathbb{N}} \leq_T D$.*

Automatic definability of $\leq_T$

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Theorem (Solecki-T., 2004)

Let D and E be basic orders. If $D \leq_T E$ then there is a **Borel map** $g : E \to D$ *which witnesses this.*

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Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω such that $\mathcal{V}$ is a P-point . If $\mathcal{U} \leq \mathcal{V}$ then there is a **continuous map** $g : \mathcal{V} \to \mathcal{U}$ *witnessing this.*

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Corollary

P-point ultrafilters have no more than continuum many Tukey-predecessors.

Ramsey expansion problem and Tukey reductions

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Theorem (Ramsey 1930, Skolem 1933)

*For every natural number k and every relation $R \subseteq \mathbb{N}^k$ there is an infinite set $M \subseteq \mathbb{N}$ such that $R \restriction M$ is $(\mathbb{N}, <)$-**canonical**.*

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Definition

*A relation $R \subseteq \mathbb{N}^k$ is $(\mathbb{N}, <)$-**canonical** on a set $M \subseteq \mathbb{N}$ if it is $\sim_{(\mathbb{N},<)}$-**invariant** on M^k, i.e., if for $(x_i : i < k), (y_i : i < k) \in M^k$, $(x_0, \dots, x_{k-1}) \sim_{(\mathbb{N},<)} (y_0, \dots, y_{k-1})$ implies $R(x_0, \dots, x_{k-1}) \Leftrightarrow R(y_0, \dots, y_{k-1})$ where we put*

$$(x_i : i < k) \sim_{(\mathbb{N},<)} (y_i : i < k)$$

if of all $i, j < k$:

$x_i < x_j \Leftrightarrow y_i < y_j$,

$x_i = x_j \Leftrightarrow y_i = y_j$,

$x_i > x_j \Leftrightarrow y_i > y_j$.

Recognizing canonical relations

Recognizing canonical relations

Proposition

There is exactly eight canonical binary relations on $\mathbb{N}$:

$$\top, \bot, =, \neq, <, >, \leqslant, \geqslant .$$

$\top$ *and* $=$ *are the only* **canonical equivalence relations** *on* $\mathbb{N}$.

Canonical equivalence relations

Theorem (Erdös-Rado 1950)

There is exactly 2^k canonical equivalence relations on $\mathbb{N}^{[k]}$:

$$(x_i : i < k) \sim_I (y_i : i < k) \Leftrightarrow (x_i : i \in I) = (y_i : i \in I),$$

for $I \subseteq \{0, ..., k-1\}$, i.e., for every equivalence relation E on

$$\mathbb{N}^{[k]} = \{(x_i : i < k) \in \mathbb{N}^k : x_0 < x_1 < \cdots < x_{k-1}\}$$

there is an infinite set $M \subseteq \mathbb{N}$ and a set $I \subseteq \{0, ..., k-1\}$ such that

$$E|M^{[k]} = \ \sim_I |M^{[k]}.$$

Higher dimensions

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Definition (Nash-Williams 1965)

A collection $\mathcal{F}$ *of finite subsets of* $\mathbb{N}$ *is a* **barrier** *if every infinite subset of* $\mathbb{N}$ *has an initial segment in* $\mathcal{F}$ *and if no two distinct elements of* $\mathcal{F}$ *are comparable under inclusion.*

Higher dimensions

Definition (Nash-Williams 1965)

A collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is a **barrier** *if every infinite subset of $\mathbb{N}$ has an initial segment in $\mathcal{F}$ and if no two distinct elements of $\mathcal{F}$ are comparable under inclusion.*

Example

- For every positive integer k, the set $[\mathbb{N}]^k = \{F \subseteq \mathbb{N} : |F| = k\}$ is a barrier.
- The family $\mathcal{S} = \{F \subseteq \mathbb{N} : |F| = \min(F) + 1\}$ is a barrier of infinite rank.

Barriers are Ramsey

Theorem (Nash-Williams, 1965)

For every barrier $\mathcal{F}$ on $\mathbb{N}$, every positive integer p, and every

$$f : \mathcal{F} \to \{0, 1, ...p - 1\}$$

there is an infinite set $M \subseteq \mathbb{N}$ such that f is constant on the restriction $\mathcal{F} \restriction M$.

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Theorem (Pudlak-Rödl, 1982)

For every equivalence relation E on some **barrier** *$\mathcal{B}$ on $\mathbb{N}$ there is an infinite set $M \subseteq \mathbb{N}$ and an* **internal irreducible** *mapping φ on $\mathcal{B} \restriction M$ such that $E \restriction (\mathcal{B}|M) = E_{\varphi}$.*

Theorem (T., 2012)

Let $\mathcal{V}$ be a **selective ultrafilter** *on $\mathbb{N}$ and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $\mathcal{U} \leq_T \mathcal{V}$. Then $\mathcal{U}$ is Rudin-Keisler isomorphic to a countable Fubini power of $\mathcal{V}$.*

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Corollary

Selective ultrafilters are **Tukey minimal** *members of $\beta\mathbb{N} \setminus \mathbb{N}$.*

Sketch of proof

Suppose $\mathcal{V} \geq_T \mathcal{U}$ with $\mathcal{V}$ selective and $\mathcal{U}$ non-principal.

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Suppose $\mathcal{V} \geq_T \mathcal{U}$ with $\mathcal{V}$ selective and $\mathcal{U}$ non-principal.
By the automatic continuity of Tukey connections in this context, there is a continuous monotone map

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mapping $\mathcal{V}$ to a generating set of $\mathcal{U}$.

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mapping $\mathcal{V}$ to a generating set of $\mathcal{U}$.Define

$$f_1 : \mathbb{N}^{[\infty]} \to \mathbb{N}$$

by $f_1(M) = \min f(M)$.
Then f_1 is also continuous so restricting f_1 we may assume that there is a barrier $\mathcal{B}$ on $\mathbb{N}$ such that for every $s \in \mathcal{B}$, the function f_1 is constant on the basic-open set $[s]$ of all infinite sets that end-extend s.

This gives us a map

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By the **selective version** of the Pudlak-Rödl theorem there is $M \in \mathcal{V}$ and $\varphi : \mathcal{B} \upharpoonright M \to M^{[<\infty]}$ such that $\varphi(s) \subseteq s$ and such that $\varphi(s) \neq \varphi(t)$ implies $\varphi(s) \nsubseteq \varphi(t)$ and $\varphi(t) \nsubseteq \varphi(s)$ and there is a one-to-one map $g : \mathrm{range}(\varphi) \to \mathbb{N}$ such that

$$(\forall s \in \mathcal{B} \upharpoonright M)\ g(\varphi(s) = f_2(s).$$

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Let $\mathcal{C}$ be the range of φ.

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Let $\mathcal{C}$ be the range of φ.
The map g gives the Rudin-Keisler equivalence between the ultrafilter $\mathcal{U}$ and the **ultrafilter** on $\mathcal{C}$ generated by

$$\{\mathcal{C} \restriction N : N \in \mathcal{V}\},$$

the **Fubini power** of the selective ultrafilter $\mathcal{V}$.

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