Basis Problems and Expansions

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What is a basis problem?

Definition

Given a pre-ordered class $(\mathcal{K}, \leq)$ *of mathematical structures of the same type, we say that* $\mathcal{K}_0 \subseteq \mathcal{K}$ *is a* **basis** *of* $\mathcal{K}$ *if for every* $K \in \mathcal{K}$ *there is* $K_0 \in \mathcal{K}_0$ *such that* $K_0 \leq K$*.*

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Suppose $\mathcal{K}_0$ is a downwards closed subclass of a given pre-ordered class $(\mathcal{K}, \leq)$ of mathematical structures.
Can one characterize $\mathcal{K}_0$ by **forbidding finitely many** *members of $\mathcal{K}$?*

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- Can one characterize in this way the class of **metrizable** compact spaces in the class of all compact spaces?

On a Problem of Formal Logic

Problem

Given a finite relational signature $\mathcal{L}$, is there a procedure that test the **validity** *of* **universal** *$\mathcal{L}$-sentences ? More generally, is there a procedure that tests* **consistency** *of sets of universal $\mathcal{L}$-sentences?*

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Definition

A k-ary relation $R \subseteq \mathbb{N}^k$ is **canonical** *if the validity of $R(x_1, x_2, ..., x_k)$ depends only on the way the usual* **ordering** *$\leqslant$ of $\mathbb{N}$ acts on x_i's.*

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Theorem (F.P. Ramsey 1930, Th. Skolem 1933)

Given a finite set T of universal relational sentences, for every positive integer m there is a positive integer n (that depends only on m and the number of relations and variables appearing in formulas of T) such that if T has a model of cardinality n or more then T has a **canonical model** *on the domain $\{1, 2, \dots, m\}$.*

The expansion problem for $\mathcal{A} = (A, R_i, f_j)_{i \in I, j \in J}$

Given a structure

$$\mathcal{A} = (A, R_i, f_j)_{i \in I, j \in J}$$

and positive integer k,

is there a **finite expansion**

$$\mathcal{A}^* = (A, R_i, f_j; S_1, ..., S_n)_{i \in I, j \in J}$$

that **captures arbitrary relation** $S \subseteq A^k$ on a **large** subset B of A, i.e.,

$$S \cap B^k = S^* \cap B$$

for $S^* \subseteq A^k$ **simply definable** in $\mathcal{A}^*$?

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$$(x_i : i < k) \sim_{(\mathbb{N}, \leqslant)} (y_i : i < k)$$

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Remark

For fixed k, the equivalence relation $\sim_{(\mathbb{N},\leqslant)}$ has finitely many equivalence classes on $\mathbb{N}^k$ which we call **atomic** $(\mathbb{N}, \leqslant)$**-canonical relations**.

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A relation $R \subseteq \mathbb{N}^k$ *is* $(\mathbb{N}, \leqslant)$-**canonical** *(or simply* **canonical***) if it is* $\sim_{(\mathbb{N},\leqslant)}$-**invariant**, *i.e., if for* $(x_i : i < k)$ *and* $(y_i : i < k)$ *in* $\mathbb{N}^k$,

$(x_0, \ldots, x_{k-1}) \sim_{(\mathbb{N},\leqslant)} (y_0, \ldots, y_{k-1})$ *implies*
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Remark

There is only finitely many canonical relation of a given arity k that can be easily identified and enumerated.
For example, there is only **eight canonical binary** relations on $\mathbb{N}$:

$$\top, \bot, =, \neq <, >, \leqslant, \geqslant .$$

$\top$ and $=$ are the only **equivalence relations** on $\mathbb{N}$ in this list.

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Theorem (F.P. Ramsey 1930)

For every positive integer k and every relation $S \subseteq \mathbb{N}^k$ there is **infinite** *subset M of $\mathbb{N}$ such that $S \cap M^{[k]} = M^{[k]}$ or $S \cap M^{[k]} = \emptyset$.*

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Corollary (F.P. Ramsey 1930, Th. Skolem 1933)

For every positive integer k, every equivalence class $\Theta \subseteq \mathbb{N}^k$ of $\sim_{(\mathbb{N},\leqslant)}$ and every relation $S \subseteq \mathbb{N}^k$ there is **infinite** *subset M of $\mathbb{N}$ such that $S \cap M^k = \Theta \cap M^k$ or $S \cap \Theta \cap M^k = \emptyset$.*

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Corollary (F.P. Ramsey 1930, Th. Skolem 1933)

For every positive integer k relation $S \subseteq \mathbb{N}^k$ there is **infinite** *subset M of $\mathbb{N}$ such that $S \cap M^k$ is canonical, i.e., equal to a union of a set of equivalence classes of $\sim_{(\mathbb{N},\leqslant)}$ restricted to M^k.*

Canonical equivalence relation on $\mathbb{N}^{[k]}$.

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Theorem (P. Erdös and R. Rado 1950)

For every positive integer k there is exactly 2^k canonical equivalence relations on $\mathbb{N}^{[k]}$, one for each subset $I \subseteq \{0, ..., k-1\}$:

$$(x_i : i < k) \sim_I (y_i : i < k) \Leftrightarrow (x_i : i \in I) = (y_i : i \in I),$$

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In other words,
for every equivalence relation E on $\mathbb{N}^{[k]}$ there is infinite $M \subseteq \mathbb{N}$ and $I \subseteq \{0, \dots, k-1\}$ such that

$$E|M^{[k]} = \ \sim_I |M^{[k]}.$$

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Fix a positive integer k. A mapping

$$f : \mathbb{N}^{[k]} \to \mathbb{N}$$

is **regressive** if

$$f(x_0, x_1, ..., x_{k-1}) < x_0 \text{ for all } (x_0, x_1, ..., x_{k-1}) \in \mathbb{N}^{[k]} \text{ with } x_0 > 0.$$

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Corollary

For every positive integer k and regressive $f : \mathbb{N}^{[k]} \to \mathbb{N}$ there is infinite $M \subseteq \mathbb{N}$ such that the restriction of f to $M^{[k]}$ is **min-constant** *i.e., for $(x_0, x_1, ..., x_{k-1}), (y_0, y_1, ..., y_{k-1}) \in M^{[k]}$,*

$$f(x_0, x_1, ..., x_{k-1}) = f(y_0, y_1, ..., y_{k-1}) \text{ whenever } x_0 = y_0.$$

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This gives us the expansion$(\mathbb{Q}, \leqslant, <')$ which generates the following equivalence relation on any finite power $\mathbb{Q}^k$:

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Theorem (R. Laver 1970)

For every positive integer k, *every equivalence class* Θ *of* $\sim_{(\mathbb{Q},\leqslant,<')}$ *and every* $S \subseteq \mathbb{Q}^k$ *there is* $M \subseteq \mathbb{Q}$ *such that*

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Corollary

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and such that $S \cap M^k$ *is* $(\mathbb{Q},\leqslant,<')$*-canonical, i.e., a union of equivalence classes of* $\sim_{(\mathbb{Q},\leqslant,<')}$.

Which classes of $\sim_{(\mathbb{Q},\leqslant,<')}$ are essential?

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A class Θ *of* $\sim_{(\mathbb{Q},\leqslant,<')}$ *on some power* $\mathbb{Q}^k$ *is* **essential** *if* $\Theta \cap M^k \neq \emptyset$ *for all* $M \subseteq \mathbb{Q}$ *such that* $(M,\leqslant) \cong (\mathbb{Q},\leqslant)$.

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Problem

For every positive integer k, classify the essential equivalence classes of $\sim_{(\mathbb{Q},\leqslant,<')}$ on $\mathbb{Q}^k$ or, equivalently, on $\mathbb{Q}^{[k]}$.

Notation: $\mathbb{Q}^{[k]} = \{(x_i : i < k) \in \mathbb{Q}^k : x_0 < x_1 < \cdots < x_{k-1}\}$.

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Theorem (D. Devlin 1979)

For every positive integer k there exist exactly $t_k = \tan^{2k-1}(0)$ essential equivalence classes of $\sim_{(\mathbb{Q},\leqslant,<')}$ on $\mathbb{Q}^{[k]}$.

Here (t_k) is the well-known sequence of **tangent numbers** which starts as $t_1 = 1,\ t_2 = 2,\ t_3 = 16,\ t_4 = 272,\ \ldots.$.

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For each positive integer k let D_k be the equivalence relation on $\mathbb{Q}^{[k]}$ whose equivalence classes are the equivalence classes of $\sim_{(\mathbb{Q},\leqslant,<')}$ on $\mathbb{Q}^{[k]}$.

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Theorem (R. Laver 1970, D. Devlin 1979)

For every positive integer k and every equivalence relation E on $\mathbb{Q}^{[k]}$ with **finitely many classes** *there is $M \subseteq \mathbb{Q}$ such that $(M,\leqslant) \cong (\mathbb{Q},\leqslant)$ and such that $E|M^{[k]}$ is* **coarser** *than $D_k|M^{[k]}$, and so in particular has no more than t_k classes.*

A 'Problem of Formal Logic' behind

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Let $\mathcal{L} = \{\in\}$. The **Zermelo-Fraenkel** axiomatic system, ZF, is the set of $\mathcal{L}$-sentences that express the familiar axioms about sets:

1. Axiom of Estensionality,
2. Empty Set Axiom,
3. Axiom of Foundation,
4. Pairing Axiom,
5. Union Axiom,
6. Power-set Axiom,
7. Axiom Schema of Separation,
8. Axiom Schema of Replacement,
9. Axiom of Infinity

If we add to ZF the **Axiom of Choice**, AC, we get the ZFC axiomatic system.

Consequences of AC

- $(\forall x)(\exists f : x \to \bigcup x)(\forall y \in x)(y \neq \emptyset \to f(y) \in y)$.
- Every set can be well-ordered.
- Zorn's Lemma.
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If ZF is consistent then so is ZFC.

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Does BPI imply AC?

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A **code** of a basic open set of $\{0,1\}^{\omega\times\omega}$ is a finite partial function $p : \omega \times \omega \to \{0,1\}$. Thus, the basic-open set coded with p is the set $[p] = \{x \in \{0,1\}^{\omega\times\omega} : x \restriction dom(p) = p\}$.

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The **Cohen Symmetric Model** is the model N is formed by elements of $M[c]$ that are **invariant** under a natural filter of groups of permutations of the first coordinate of the set $\omega \times \omega$.

Cohen's Symmetric Model

Start with a countable transitive model M of ZF and going to a submodel assume that every element of M is Gödel-constructible. Then ω, the first infinite ordinal, belongs to M and therefore the version $\{0,1\}^{\omega\times\omega}$ of the Cantor space.

A **code** of a basic open set of $\{0,1\}^{\omega\times\omega}$ is a finite partial function $p : \omega \times \omega \to \{0,1\}$. Thus, the basic-open set coded with p is the set $[p] = \{x \in \{0,1\}^{\omega\times\omega} : x \restriction dom(p) = p\}$.

A collection $\mathcal{D}$ of finite partial functions from $\omega \times \omega$ into $\{0,1\}$ **codes** the open set $[\mathcal{D}] = \bigcup_{p\in\mathcal{D}}[p]$.

Since there is only countably many dense open subsets of $\{0,1\}^{\omega\times\omega}$ coded in M their intersection is not empty.

Pick $c : \omega \times \omega \to \{0,1\}$ from this intersection.

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The **Cohen Symmetric Model** is the model N is formed by elements of $M[c]$ that are **invariant** under a natural filter of groups of permutations of the first coordinate of the set $\omega \times \omega$.Thus, $M \subseteq N \subseteq M[c]$.

Properties of the Cohen Symmetric Model

For $n \in \omega$, let $c_n : \omega \to \{0, 1\}$ be defined by $c_n(m) = c(n, m)$.
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The proof of this result relies on a deep combinatorial theorem of Halpern-Läuchli that is also the key ingredient of the solution to the expansion problem for $(\mathbb{Q}, \leqslant)$ discussed above.

The Halpern-Läuchli theorem

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*For $k \in \omega$ and $x \in T$, a subset X of T is k-x-**dense** if it dominates every node of $T[x]$ of height k.*
*A subset X of T is **somewhere dense** if it is k-x-dense for some $x \in T$ and some k **above** the height of x.*

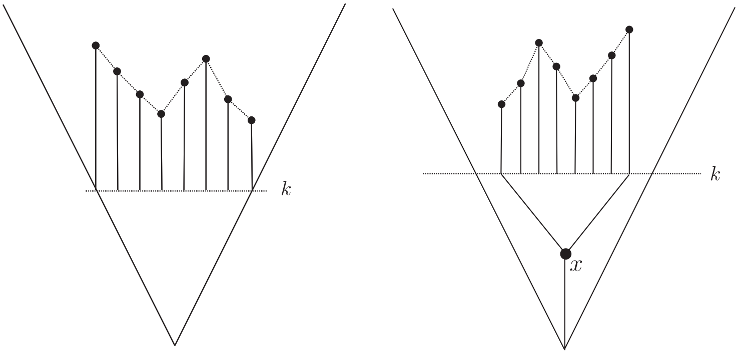

Figure: A *k*-dense set and a *k*-*x*-dense set

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A grid $\prod_{i<d} X_i$ is **somewhere dense** if it is k-$\bar{x}$-dense for some $\bar{x} \in \prod_{i<d} T_i$ and k **above the height of** x_i **for all** $i < d$.

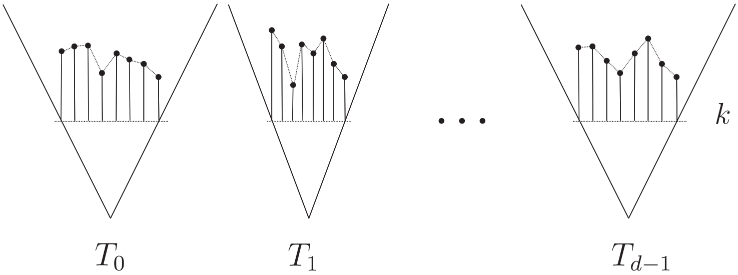

Figure: A k-dense grid

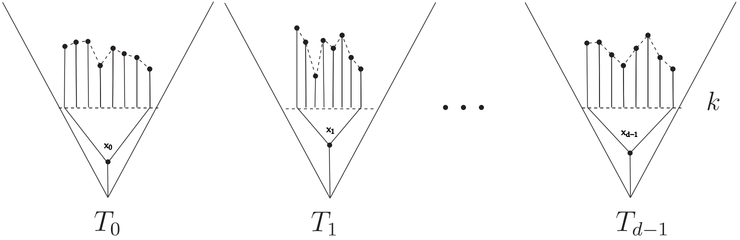

Figure: A k-$\bar{x}$-dense grid

The Halpern-Läuchli Theorem

Theorem (J.D. Halpern and H. Läuchli, 1966)

For every finite sequence $T_0, T_1, ..., T_{d-1}$ *of rooted finitely branching trees of height* ω *with no terminal nodes and for every positive integer* p *there is a positive integer* n *such that for every* n*-dense grid* $\prod_{i<d} X_i$ *of* $\prod_{i<d} T_i$ *and every*

$$f : \prod_{i<d} X_i \to \{0, 1, ..., p-1\}$$

there is a somewhere dense sub-grid $\prod_{i<d} Y_i \subseteq \prod_{i<d} X_i$ *on which the function* f *is constant.*

Strong subtrees

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A **strong subtree** of T of height $1 \leq k \leq \omega$ is a rooted subtree U of height k for which we can find increasing sequence $(n_i)_{i<k}$ of non-negative integers such that

- $U(i) \subseteq T(n_i)$ for all $i < k$,
- for every $i < k - 1$, every $s \in U(i)$, and every immediate successors t of s in T there is exactly one $u \in U(i+1)$ extending t.

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For $1 \leq k \leq \omega$, let

$$\mathrm{Str}^k(T) = \{U : U \text{ a strong subtree of } T \text{ of height } k\}.$$

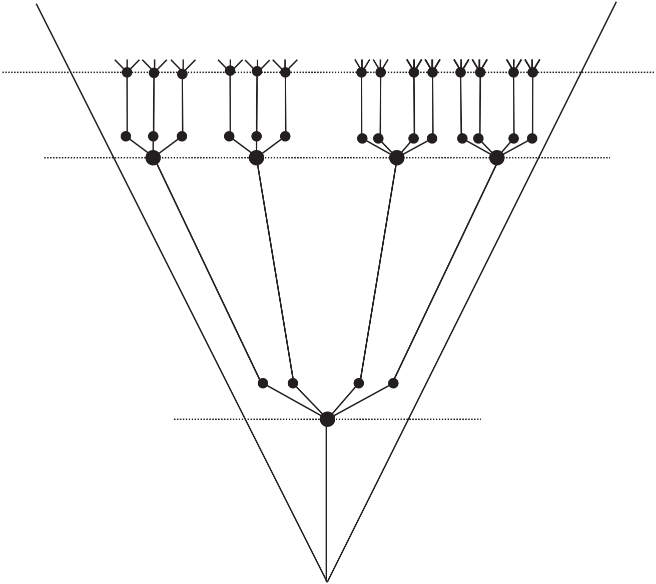

Figure: The tree T and one of its strong subtrees U of height 3.

The Milliken theorem

Theorem (K. Milliken, 1979)

For every rooted finitely branching tree T of height ω with no terminal nodes, every positive integers k and p, and every mapping

$$f : \mathrm{Str}^k(T) \to \{0, 1, 2, ..., p - 1\}$$

there is a strong subtree U of T of height ω such that f is constant on $\mathrm{Str}^k(U)$.

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Corollary (F.P. Ramsey, 1930)

For every positive integers k and p and every

$$f : \mathbb{N}^{[k]} \to \{0, 1, 2, ..., p-1\}$$

there is infinite $M \subseteq \mathbb{N}$ such that f is constant on $M^{[k]}$.

An expansion problem of the random graph

Fix a representation $\mathcal{R} = (\mathbb{N}, R)$ of the **random graph**, i.e.,

- R is a symmetric irreflexive binary relation on $\mathbb{N}$,
- every finite graph is isomorphic to an induced subgraph of $\mathcal{R}$,
- every isomorphism between two induced subgraphs extends to an automorphism of $\mathcal{R}$.

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Problem (Expansion problem)

Fix a positive integer k. Is there an expansion of the random graph $(\mathbb{N}, R)$ that canonizes sub-relations of $\mathbb{N}^{(k)}$ when restricted to a copy of an **arbitrarily large finite** *induced subgraph of $\mathcal{R}$.?*

Notation: $\mathbb{N}^{(k)} = \{(x_0, x_1, ..., x_{k-1}) \in \mathbb{N}^k : x_i \neq x_j \text{ for } i \neq j\}$.

The Nešetřil-Rödl expansion of the random graph

We explain the Nešetřil-Rödl expansion in the case of the random graph $\mathcal{R} = (\mathbb{N}, R)$ which however works for many other countable homogeneous structures.

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Definition

Fix a linear ordering $\leqslant^*$ *on* $\mathbb{N}$ *such that the expanded structure* $(\mathcal{R}, \leqslant^*) = (\mathbb{N}, R, \leqslant^*)$ *has the following properties*

- *every isomorphism between two finite substructures of* $(\mathcal{R}, \leqslant^*)$ *extends to an automorphism of* $(\mathcal{R}, \leqslant^*)$,
- *every* **finite ordered graph** *is isomorphic to a substructure of* $(\mathcal{R}, \leqslant^*)$.

Fix a finite graph graph $\mathbf{A} = (V_A, E_A)$ and assume that its vertex-set V_A is equal to $\{0, 1, ...k - 1\}$ for some positive integer k.

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The graph $\mathbf{A}$ naturally leads to the following **atomic** $\mathcal{R} = (\mathbb{N}, R)$**-canonical** k**-ary relation**

$$C_{\mathbf{A}} = \{\bar{x} \in \mathbb{N}^{(k)} : (\forall i < j < k)[(i,j) \in E_A \leftrightarrow (x_i, x_j) \in R]\}.$$

Thus $C_{\mathbf{A}}$ is simply the set of all embeddings of $\mathbf{A}$ into the random graph $\mathcal{R}$, i.e.,

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The Nešetřil-Rödl expansion $(\mathcal{R}, \leqslant^*) = (\mathbb{N}, R, \leqslant^*)$ splits each $C_{\mathbf{A}}$ into its own **atomic canonical relations**

$$C_{\mathbf{A}}^{\sigma} = \{\bar{x} \in C_{\mathbf{A}} : (\forall i < j < k)\; x_{\sigma(i)} <^* x_{\sigma(j)}\},$$

where σ is an arbitrary permutation of $\{0, 1, ...k-1\}$.

Theorem (F.G. Abramson and L. Harrington 1978, J. Nešetřil and V. Rödl 1977)

- *The Nešetřil-Rödl expansion* $(\mathcal{R}, \leqslant^*) = (\mathbb{N}, R, \leqslant^*)$ *solves the expansion problem for the random graph* $\mathcal{R} = (\mathbb{N}, R)$*, i.e., for every positive integer* k *for every relation* $S \subseteq \mathbb{N}^{(k)}$ *there is an* **arbitrarily large finite** *induced subgraph* (B, R) *of* $\mathcal{R} = (\mathbb{N}, R)$ *such that* $S \cap B^{(k)}$ *is* $(\mathcal{R}, \leqslant^*)$*-canonical.*

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- *In particular, for every finite graph* $\mathbf{A} = (V_A, E_A)$ *such that* $V_A = \{0, 1, ...k-1\}$ *of some* k *and every permutation* σ *of* $\{0, 1, ...k-1\}$ *the relation* $C^{\sigma}_{\mathbf{A}} \subseteq \mathbb{N}^{(k)}$ **is Ramsey**, *i.e., for every relation* $S \subseteq \mathbb{N}^{(k)}$ *there is an* **arbitrarily large finite** *induced subgraph* (B, R) *of* $\mathcal{R} = (\mathbb{N}, R)$ *such that*

$$S \cap C^{\sigma}_{\mathbf{A}} \cap B^{(k)} = \emptyset \text{ or } S \cap B^{(k)} \supseteq C^{\sigma}_{\mathbf{A}} \cap B^{(k)}.$$

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The class of finite graphs has the **ordering property**, *i.e., for every finite graph* (A, E_A) *there is a* **finite ordered graph** $(B, E_B, <_B)$ *such that for every linear ordering* $<'$ *on* A *there is an embedding from* $(A, E_A, <')$ *into* $(B, E_B, <_B)$.

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Corollary

For every finite graph $\mathbf{A} = (V_A, E_A)$ *with* $V_A = \{0, 1, ...k-1\}$ *for some* k *and every permutation* σ *of* $\{0, 1, ...k-1\}$ *the relation* $C^\sigma_{\mathbf{A}} \subseteq \mathbb{N}^{(k)}$ **is essential**, *i.e, for every sufficiently large induced subgraph* (B, R) *of the random graph* $\mathcal{R} = (\mathbb{N}, R)$ *the intersection* $C^\sigma_{\mathbf{A}} \cap B^{(k)}$ *is not empty.*

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Fix a finite graph $\mathbf{A} = (V_A, E_A)$ *with* $V_A = \{0, 1, \ldots k-1\}$ *for some* k. *Let* $\mathcal{E}_{\mathbf{A}}$ *be the equivalence relation on the set* $\mathcal{E}_{\mathbf{A}}$ *of all embeddings from* $\mathbf{A}$ *into the random graph* $\mathcal{R}$ *whose classes are the sets* $C_{\mathbf{A}}^{\sigma}$ *where* σ *runs over all permutations of* $\{0, 1, \ldots k-1\}$. *Then*

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- *All* $k!$ *classes of* $\mathcal{E}_{\mathbf{A}}$ *are realized on every sufficiently large finite induced subgraph or the random graph* $\mathcal{R}$.
- *For every equivalence relation* $\mathcal{E}$ *on* $C_{\mathbf{A}}$ *with finitely many equivalence classes there is an arbitrarily large induced subgraph* B *of* $\mathcal{R}$ *such that the restriction* $\mathcal{E}$ *on the set* $C_{\mathbf{A}} \cap B^{(k)}$ *of all embeddings from* $\mathbf{A}$ *into* (B, R) *is* **coarser** *then the restriction of the canonical equivalence relation* $\mathcal{E}_{\mathbf{A}}$ *on the same set.*

Dynamics of the group $\mathrm{Aut}(\mathcal{R})$

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Theorem (A. Kechris, V. Pestov and S. T., 2005)

Let $\mathrm{LO}(\mathbb{N})$ *be the compact space of all linear orderings on* $\mathbb{N}$, *the vertex-set of the random graph* $\mathcal{R}$. *Let* $G = \mathrm{Aut}(\mathcal{R})$ *and let* $\alpha : G \times \mathrm{LO}(\mathbb{N}) \to \mathrm{LO}(\mathbb{N})$ *be the natural continuous action. Then:*

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- *The action* α *is* **minimal**, *i.e., all of its orbits are dense in* $\mathrm{LO}(\mathbb{N})$.
- *The action* α *is* **universal**, *i.e., for any other minimal continuous action* $\beta : G \times X \to X$ *on a compact space* X *there is a continuous onto map* $\pi : \mathrm{LO}(\mathbb{N}) \to X$ *that commutes with the actions.*

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