REFLECTIONS ON MATHEMATICAL THINKING AND ENGAGING WITH MATHEMATICS THOUGHTFULLY

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Mathematics is a Thoughtful Enterprise

- Richard Courant: “Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection.”
- I interpret this as follows: **mathematics is a thoughtful activity infused with human agency and a quest for beauty** <tie to final Q?>
- This stands in sharp contrast to how many students perceive mathematics

Students Perceive Mathematics as …

- Dull and boring
- Not something to be understood: Arbitrary rules and procedures to memorize and apply mechanically
- Lacking connection to people or relevance to the “real world”

Dull and Boring?

From 19th C Russian memoir re mathematics lessons < eerily similar to my calculus experience nearly 150 years later> :

Walking into the classroom with his eyes lowered to the ground, he would come up to the blackboard, stand with his back to the class, and begin his lecture in a quiet, monotonous drone or almost in a whisper, reading from his notes and making various calculations and sketching mathematical figures on the blackboard…. For the most part, the cadets slept through his classes.

Karp, Alexander. 2007. We all meandered through our schooling…: Notes on Russian mathematics education in the early nineteenth century.

Not something to be understood & Lacking connection or relevance

Third National Assessment of Educational Progress (NAEP, 1983):

"An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?"

"An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?"

- What percent of 13-year-olds would you expect to successfully answer this constructed-response question?
- Only 24% gave a correct answer of 32
- 25% gave no answer
- About 30% gave an answer of 31.33 or 31 1/3;
- Some gave 31R12, 12 or 31
- These results pointed to a lack of sense making in Ss’ problem solving.

Reflections on Mathematical Thinking
Outline of this lecture

- Introductory remarks: Why mathematical thinking?
- Problem solving and mathematical thinking
- Helping teachers provide opportunities for mathematical thinking
- Concluding remarks

Mathematical Problem Solving

- The importance of problems and problem solving in mathematics education has long been recognized by mathematicians and educators [Paul Halmos: "…the mathematician's main reason for existence is to solve problems … what mathematics really consists of is problems and solutions."
- Problem tasks are the backbone of classroom mathematics instruction across the globe, and the generation of problem solutions is a common focus of student activity in mathematics classrooms.
- Though problem solving has long been a major topic of emphasis in the mathematics education community, ***the classroom reality is often far different than the envisioned possibilities***!

Problem Solving in Mathematics Class: The Ritual

- Ts provide a problem and Ss expect that the problem will require them to use/practice some method the teacher has recently taught
- If that method does not quickly result in a solution, then they ask the teacher for help
- Ss too often appear to lack a sense of personal agency or autonomy to persist in trying to solve problems when the method is not obvious
- Ts unsure how to assist in some way other than giving the answer or demonstrating the method

Typical Student Beliefs about Mathematics Based on Their Problem Solving Experience (Schoenfeld & other sources)

- Mathematics problems come only from teachers and textbooks
- Mathematics problems have one and only one right answer
- There is only one correct way to solve any mathematics problem—usually a procedure the teacher has recently demonstrated
- Just do it the right way and don’t worry if the answer makes sense
- Students who are “good” at mathematics should be able to solve any assigned problem in 5 minutes or less
- If it takes more than 5 minutes to solve a problem, you can’t do it

Mathematical Problem Solving as Thoughtful Activity: Polya's Heuristics

- Through his writings about mathematical problem solving, George Polya explicated a view of mathematics as a human activity concerned about ideas, thinking and creation
- ***How to Solve It***: Problems may take time to solve; the method may not be obvious; heuristic suggestions can help; students can be agents of their own problem solving
- His writings had great influence on me as a doc S during the 1970s, an era of great interest in non-routine problem solving

Mathematical Problem Solving as Thoughtful Activity: Polya's Heuristics

- My dissertation focused on one of his heuristic suggestions: If you are stuck in solving a certain problem … **Think of a related problem**
- I was teaching 7th and 8th grade Ss at the time and was curious about how Ss perceived relatedness among mathematics problems
- Findings:
  - Students differed in their perceptions of problem relatedness and that the differences were linked to mathematical problem-solving performance
  - Students thought about problem relatedness in different ways, not all of which would likely make Polya's heuristic advice useful

  <see Student perceptions of relatedness among mathematical verbal problems, JRME 1979>

The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

Sense Making and the Solution of Division Problems Involving Remainders: An Examination of Middle School Students' Solution Processes and Their Interpretations of Solutions, JRME 1993

- Grade 6-8 Ss: show their work and explain solution
- 43% gave 14 as answer; about 20% gave 13, 13.5, 13 ½, or 13 R 20
- More than 30% gave incorrect [often unreasonable] numerical answer due to errors in executing long division
- More than half of explanations were purely procedural; Disconnect between math calculation and sensibleness of answer was evident for many students
- BUT about 1% gave intriguing responses for 13 ½ **: "You need 13 buses and 1 van [or minibus] for the others"**

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How about Problem Posing?

Write and solve problems that…

can be solved using the

same division statement

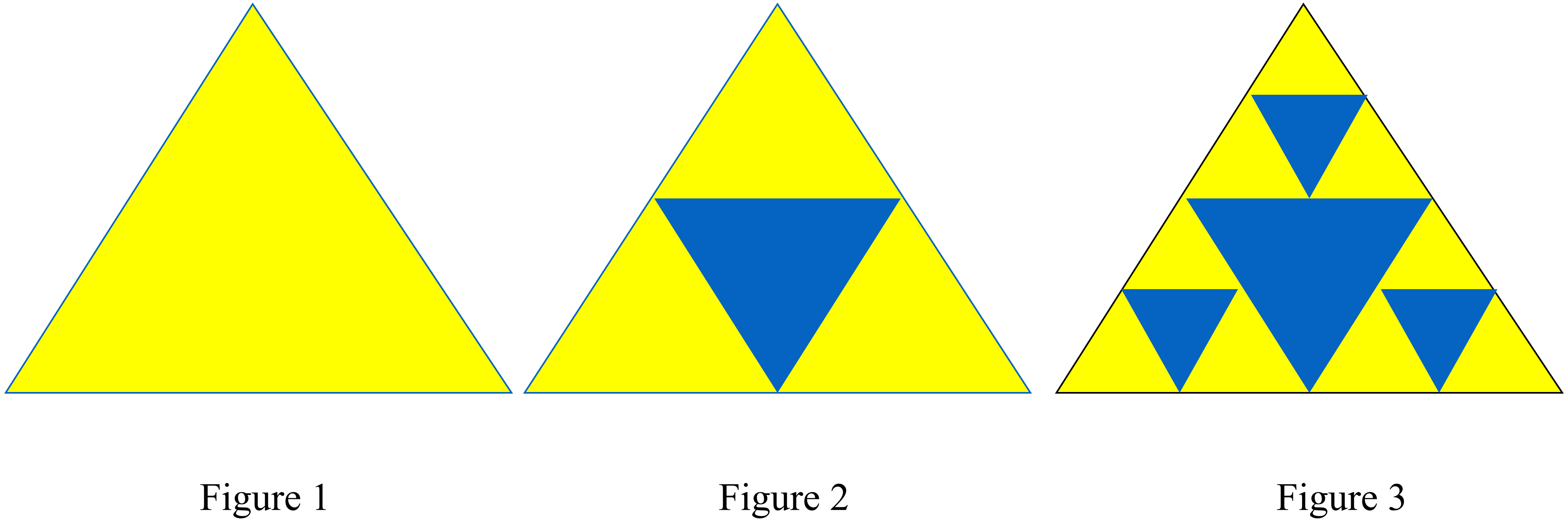
540 ÷ 40 = ???

but have ***different solutions.***

Can you propose two?

Are there others?

Each side of the triangle in Figure 1 is 1 unit long. Successive Figures are formed by connecting midpoints of the sides of yellow triangles to obtain smaller blue triangles. What problems are suggested?



Modified from original version in NCTM, *Figure This!*

Show that the product of 4 consecutive integers is always 1 less than a perfect square

Open Version:
Consider the product of 4 consecutive integers. Make some conjectures.

Conjectures about the product of 4 consecutive integers

…never negative

… always even/never odd

… divisible by 3

… divisible by 6

… divisible by 12

… divisible by 24

… one less than a perfect square

More Problem-Posing Opportunities ala Brown & Walter "What-if-not?"

What if 3 consecutive integers? Or 5 or 6?
Or N?

What if consecutive even integers? Or
consecutive odd integers

Problem Posing in School Mathematics

- Jeremy Kilpatrick, 1987, *Problem formulating: Where do good problems come from?*
- Based on their school experience, students would say that virtually all mathematics problems come from teachers and textbooks.
- BUT in real life "many problems, if not most, must be created or discovered by the solver, who gives the problem an initial formulation"
- "**Problem formulating should be viewed not only as a goal of instruction but also as a means of instruction**…
- …**all students' school experience should include opportunities to discover and pose their own problems**."

Problem Posing as Mathematical Activity?

Georg Cantor: "In mathematics the art of proposing a question must be held of higher value than solving it."

Paul Halmos: "The best way to conduct a problem seminar is, of course, to present problems, but it is just as bad for an omniscient teacher to do all the asking in a problem seminar as it is for an omniscient teacher to do all the talking in a lecture course. **I strongly recommend that students in a problem seminar be encouraged to discover problems on their own**."

Problem Posing Can Disrupt the Classroom Ritual

- Ts provide a problem and Ss expect that the problem will require them to use/practice some method the teacher has recently taught Ss can generate problems for themselves and peers
- If that method does not quickly result in a solution, then they ask the teacher for help
- Ss too often appear to lack a sense of personal agency or autonomy to persist in trying to solve problems when the method is not obvious Ss can persist with a belief in their own efficacy
- Ts unsure how to assist in some way other than giving the answer or demonstrating the method Ts can allow Ss to exhibit personal agency

Affective Benefits of Problem Posing

- Research in past 30 years on affective aspects of problem posing supports a speculation I made in my 1994 paper, *On Mathematical Problem Posing*: **Providing students with problem-posing experiences is likely to improve their disposition toward mathematics**
- Meta-analyses of these research findings suggest a positive association between students' experiences with problem posing and their dispositions toward mathematics, which includes beliefs about, attitudes toward and interest in mathematics

Helping Teachers Provide Opportunities for Mathematical Thinking

- QUASAR
- COMET
- ASTEROID
- ESP
- BIFOCAL

QUASAR: Some Lessons Learned

- Tasks vary in their likelihood of evoking students' thoughtful engagement (cognitive demand: low vs high)
- The cognitive demands of a task can evolve during instruction
- Consistent engagement with high-level tasks leads to the greatest learning gains for students <similar finding later in TIMSS Video Study, NCES, 2003>
- BUT enacting high-level mathematical tasks well is not easy

Tasks Analysis Guide

Lower-Level Demands	Higher-Level Demands
<u>Memorization</u> • involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.	<u>Procedures With Connections</u> • focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
<u>Procedures Without Connections</u> • are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or` placement of the task. • require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or explanations that focuses solely on describing the procedure that was used.	<u>Doing Mathematics</u> • require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • require students to explore and understand the nature of mathematical concepts, processes, or relationships. • demand self-monitoring or self-regulation of one's own cognitive processes. • require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

<u>Figure 2. 3</u> Characteristes of mathematical instructional tasks*.

*These characteristics are derived from the work of Doyle on academic tasks (1988), Resnick on high-level thinking skills (1987), and from the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, & Henningsen, 1996; Stein, Lane, and Silver, 1996).

Stein, Smith, Henningsen, & Silver, 2009, p.6

Features of Cognitively Demanding Mathematics Tasks

- **Doing mathematics**
  - complex, nonalgorithmic thinking
  - problem solving and reasoning
  - exploration, inference and extrapolation

- **Procedures with connections**
  - using procedures to develop deeper understanding
  - solution connected to underlying mathematical ideas
  - problem elements represented in multiple ways (e.g., visual diagrams and equations)

Mathematics Tasks & Teachers

Tasks are important, but teachers also matter!

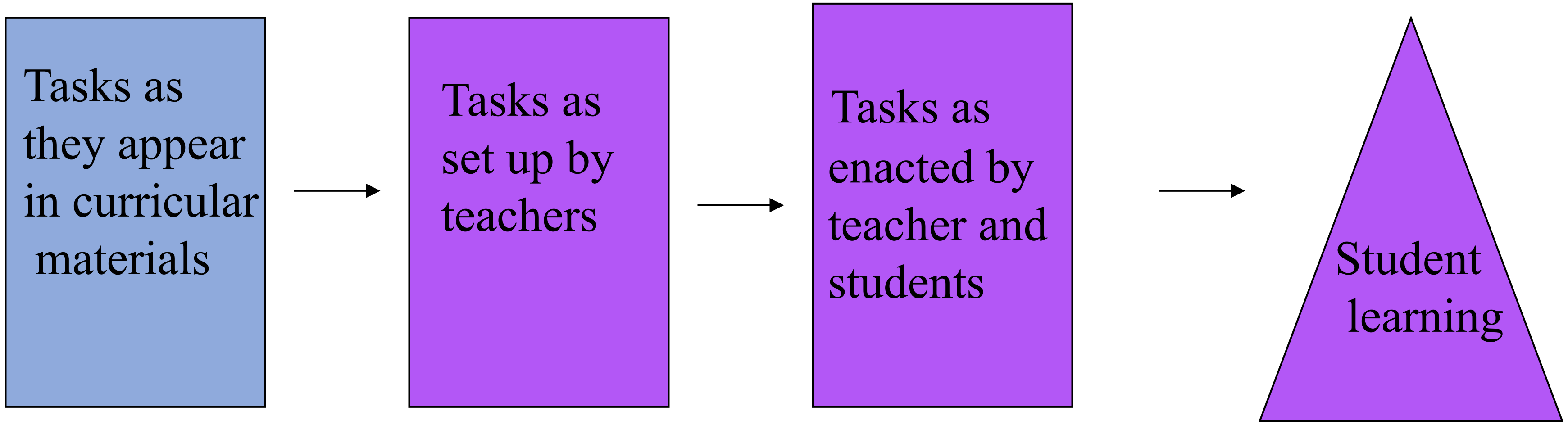
Teacher actions and reactions …

influence the *nature and extent of student engagement* with challenging tasks,

and

affect *students' opportunities to learn* from and through task engagement.

The Mathematical Tasks Framework

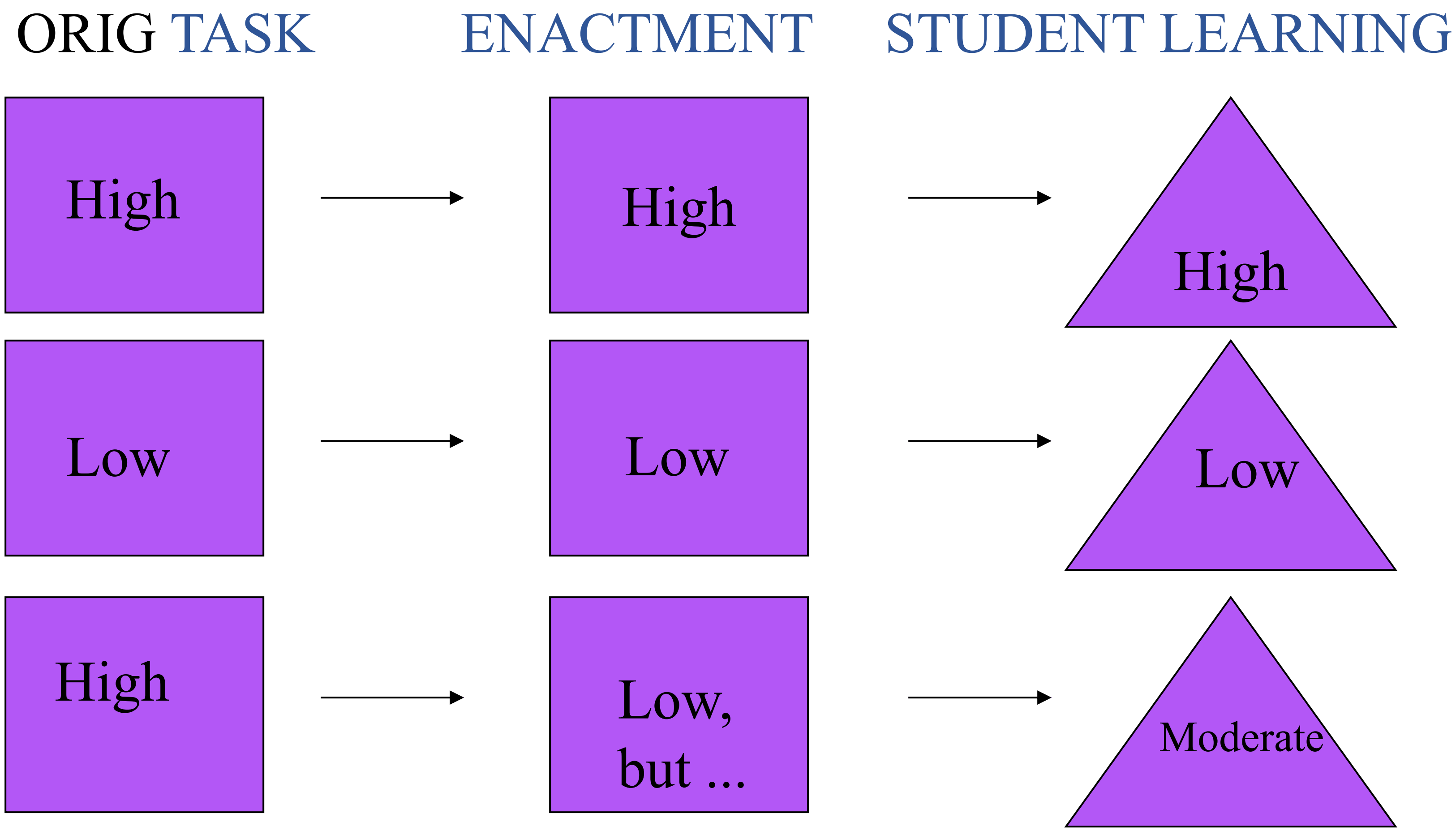


Stein, Grover & Henningsen (1996)

Smith & Stein (1998)

Stein, Smith, Henningsen & Silver (2009)

Task Demands and Enactment Quality Both Affect Student Learning



Teaching with Cognitively Demanding Mathematics Tasks

Teachers must decide "what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge."

NCTM, 2000, p.19

Some Challenges Facing Teachers

- **Resisting the persistent urge to tell and to direct; *allowing time for student thinking***
- **Knowing when/how to ask questions and to provide information *to support rather than replace* student thinking**
- **Helping students accept the challenge of solving worthwhile problems and *sustaining their engagement* at a high level**

Mathematics Teachers Can Get Better at Creating Opportunities for Mathematical Thinking

- The MTF provides a useful framework for Ts to …
- develop a more nuanced understanding of their role as mediator of Ss’ interaction with mathematical tasks;
- increase the level of cognitive demand of math tasks used in their classrooms;
- enhance their repertoire of useful instructional strategies to maintain high-level cognitive demands; and
- be more effective in maintaining high-level cognitive demands during classroom lessons.

William A. Brownell

- Ph.D. Univ of Chicago 1926
- His multi-decade research on human learning, particularly arithmetic, demonstrated that *understanding*, not simply repetition, could be the basis for children's mathematical learning <in contrast to the behaviorist views of E. L. Thorndike>
- As a doc student I read and was inspired by his work
- My Collegiate Professorship @ UM is named for him <briefly @ UM prior to Duke & UC-Berkeley>

William A. Brownell

“Arithmetic has both a mathematical aim and a social aim. To be intelligent in quantitative situations *children must see sense in the arithmetic they learn*. Hence, instruction must be meaningful and must be organized around the ideas and relations inherent in arithmetic as mathematics. But they must also have experiences in using the arithmetic they learn in ways that are significant to them at the time of learning... **We have no choice**; we cannot emphasize one of the two aims, to the exclusion of the other. Both aims are essential … and both are attainable.”

(W. A. Brownell, 1954,p. 5)

"We have no choice"
Engaging with Mathematics Thoughtfully

- ~~Dull and boring~~ **Engaging and thought-provoking**
- ~~Not something to be understood: Arbitrary rules and procedures to memorize and apply mechanically~~ **Possible to understand**
- ~~Lacking connection to people or relevance to the "real world"~~ **Something WE can do and WE can use**