Representation Theory for Risk On Markowitz-Tversky-Kahneman Topology: Risk Torsion and Loss Aversion

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Introduction

- What does the paper do?
  This paper fills a gap in the literature on behavioural economics by introducing a representation theory of decision making in the presence of risk and uncertainty on locally compact groups in a topological manifold.
- What is the motivation for the paper?
  The theory is motivated by geometry of the utility function introduced by [Markowitz, 1952] (M) and [Kahneman and Tversky, 1979] and [Tversky and Kahneman, 1992, pg. 303] (TK)
- Applications of the theory include characterization of risk attitudes with infinitesimal generators, estimates of local and global implied loss aversion index, trace class estimates of implied loss aversion, perceptions of risk, asset pricing, and introduction of eigen-payoffs in economic experiments (not included here)

The topology

- What is the topology induced by the geometry of [Markowitz, 1952] and [Kahneman and Tversky, 1979] utility functions?
  - [Markowitz, 1952] posited a utility function u, for wealth x, around the origin such that $u(x) > |u(-x)|$ and "$x = 0$ is customary wealth", *id.,* at 155. This is a *de facto* reference point for gains or losses in wealth. Each of the subject inflection points are critical points for risk dynamics with basis sets

$$U_\alpha^M = \{x|\ u(x) > |u(-x)|,\ x > 0,\ -x < 0 < x\},\ \ \alpha \in A$$

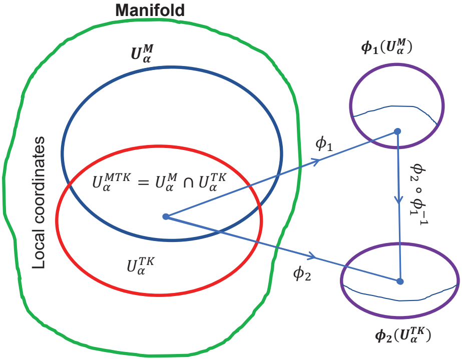
    for some index set A.
  - [Kahneman and Tversky, 1979, pg. 277] also introduced a *reference point* hypothesis. Theirs is based on "perception and judgment", and they "hypothesize that the value function $[v]$ for changes of wealth $[x]$ is normally concave above the reference point $(v''(x) < 0,\ \text{for } x > 0)$ and often convex below it $(v''(x) > 0,\ x < 0)$" [parentheses added], *id.,* at 278. The basis sets here are

$$U_\alpha^{TK} = \{x|\ u''(x) < 0, x > 0;\ u''(x) > 0, x < 0;\ -x < 0 < x\}\ \ \alpha \in A$$

- The refined topology. The seminal papers above support examination of risk dynamics for transformation groups in a neighbourhood of the origin [or critical points] which, by definition, are included in a topological manifold characterized by the following basis sets for a refined topology:
  - $U_\alpha^{MTK} = U_\alpha^M \cap U_\alpha^{TK}$. If U^M and U^{TK} are the sets of points in M or TK neighbourhoods of inflection points, then $U^{TK} \subseteq \bigcup_\alpha U_\alpha^{MTK}$ and $U^M \subseteq \bigcup_\alpha U_\alpha^{MTK}$ for index $\alpha \in A$.

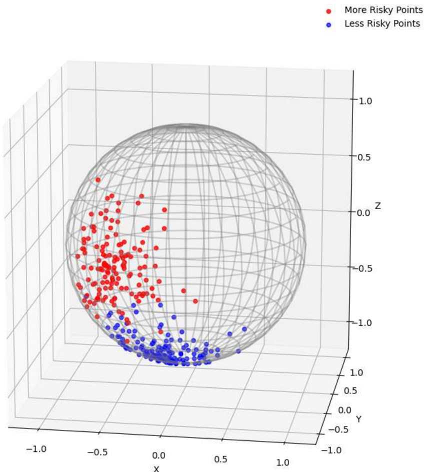
The topological manifold

Figure: Schematic manifold for MTK topology



The sketch depicts a classic manifold for the typical open sets $U_\alpha^{MTK} = U^M \cap U_\alpha^{TK}$ in the MTK topology. (U_α, ϕ_α) are coordinates in the system where ϕ_α is a continuous mapping $\phi_\alpha : U_\alpha \to \mathbb{R}^d$.

Example: 2-mean clustering on a Riemannian Manifold

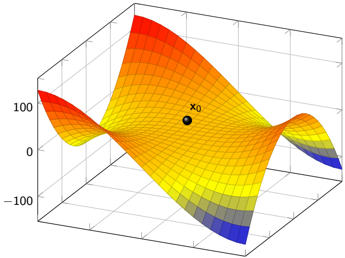


The plot depicts a K-mean (K=2) clustering on a Riemannian Manifold. It is constructed by generating data on a sphere from a Von Mises Fisher distribution (i.e., a probability distribution on a sphere) with a rotation sampled randomly from special orthogonal group SO(3). See [Miolane et al., 2020] for details, and explanation of the Python code.

How does any of this relate to "quantum" anything?

- The Gauss curvature $K(\mathbf{x}_0)$ associated to a reference point $\mathbf{x}_0$ on the topological manifold of a MTK utility hypersurface is hyperbolic
- In our setup, the curvature is characterized by a "risk torsion" operator (call it A) under expected utility theory (EUT) with trace(A)=0. This is a classic feature of the quantum group $SU(n)$ which deals with rotational symmetry.

Figure: Hyperbolic point $\mathbf{x}_0$ on utility hypersurface $u(\mathbf{x}, \mathbf{y})$



- The plot depicts "risk torsion"– the tension between risk seeking over convex, and risk aversion over concave, portions of the utility surface $u(\mathbf{x}, \mathbf{y})$.
- The domain of the risk torsion operator A is the hyperbolic portion of the surface.
- The ball is at the reference point $\mathbf{x}_0 = 0$ on the hyperbolic surface that is characterized by the Gauss curvature K($\mathbf{x}_0$) (which is a function of the tangents at $\mathbf{x}_0$). On a "planar surface" K($\mathbf{x}_0$)=0.

Towards a Lie algebra of risk operations I

Definition (Logarithmic differential operator)

A logarithmic differential operator $\ln D$ is defined for all functions u in the domain $\mathfrak{D}(D)$ of D such that

$$(\ln Du)(x) = \operatorname{sgn}(u'(x)) \ln |u'(x)|, \; u'(x) \neq 0$$

This definition is general enough to handle $u'(x) < 0$ and is undefined for $u'(x) = 0$. □

Definition (Arrow-Pratt risk operator)

Let X be a compact choice space, and $u \in C_0^2(X) \cap \mathfrak{D}(D)$ be a twice differentiable continuous utility function. Let D be the differential operator so that $(Du)(x) = u'(x)$ and $(D^2 u)(x) = u''(x)$. Then the Arrow-Pratt risk operator A for the risk measure $r(x)$ is given by

$$r(x) = (Au)(x), \qquad A = -D \ln D = -\left(\frac{D^2}{D}\right)$$

In the sequel we use A_{ra} instead of A for risk averse operations, and A_{rs} instead of A to distinguish risk seeking operations. □

Towards a Lie algebra of risk operations II

- Let $X \subset \mathbf{R}^n$ be an open space of choice vectors, i.e., n-dimensional basket of goods; G be a compact group in X; $\mathbf{x}, \mathbf{y} \in$ G; and $\mathbf{u} : G \rightarrow V \subset \mathbf{R}^n$ be a vector valued utility function. By definition (see [Michor, 1997]), G is a topological manifold, i.e. a *topological group*. Assume that V is a Lie group germ induced by G. For example V could be a (vector valued) indirect utility function , i.e., $V(\mathbf{p}, I) := \{\mathbf{x} \in \mathbf{R}^n_+ : \mathbf{p} \cdot \mathbf{x} = I\}$ for income level I, price vector $\mathbf{p}$, and consumption bundle $\mathbf{x} \in \mathbf{R}^n_+$. See [Varian, 1992, p. 99].
- Let $\mathrm{A}_{ra} = -D \ln D$ be the operator for Arrow-Pratt risk aversion (ra) described in the previous slide. The corresponding infinitesimal vectors for $\mathbf{x}, \ \mathbf{y} \in G$ are $\boldsymbol{\alpha} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{t=0}$ and $\boldsymbol{\beta} = \left(\frac{\partial \mathbf{y}}{\partial t}\right)_{t=0}$, which stem from the expansion

$$\mathbf{x} = \boldsymbol{\alpha} t + \dots \qquad \mathbf{y} = \boldsymbol{\beta} t + \dots$$

  The above gives rise to the following relationship between group operations in G and vector addition of infinitesimal vectors:

Theorem (Infinitesimal vectors of group product)

[Guggenheimer, 1977, pg. 104]
Let $\mathbf{x}, \ \mathbf{y} \in C^n(X)$ be curves in G, with infinitesimal vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The curve $\mathbf{xy}$ is differentiable and it has infinitesimal vector $\boldsymbol{\alpha} + \boldsymbol{\beta}$. □

Towards a Lie algebra of risk operations III

Theorem (Ado's theorem)

[Nathanson, 1979, pg. 202]
Every finite dimensional Lie algebra $\mathcal{L}$ of characteristic zero has a finite dimensional representation. □

Remark

A field F has characteristic 0 if for any $a \in F$ and $n \in \mathbf{N}$ $na = 0$ implies $a = 0$. For example, if the "additive identity" element of the field is 0, it is the number of times we must add the identity to get 0. See [Clark, 1971, pg. 69]. The theorem basically says, for example, that a finite dimensional Lie algebra with characteristic 0 has a representation in the matrix group GL. □

Theorem (Lie algebra of risk operation on Abelian group)

The Lie algebra $\mathcal{L}(G)$ induced by risk operations on VNM utility with support on the abelian group G is that of the antisymmetric or skew symmetric matrices $\mathcal{L}(\mathcal{O}_n)$. □

Infinitesimal generators of risk attitudes

For differentiable curves $\mathbf{x}(t)$ and $\mathbf{y}(t)$, with parameter t, i.e., one parameter group of motions, the Lie group structure for risk attitudes associated to $\mathbf{u}(\mathbf{x},\mathbf{y})$, i.e., the infinitesimal generator of risk attitudes, is determined by:

Risk aversion

$$\begin{aligned}(\mathrm{A}_{ra}\mathbf{u})_k &= ((-\mathrm{DlnD})\,\mathbf{u}\,(\mathbf{x},\mathbf{y}))_k = (\boldsymbol{\alpha\beta})_k \\ &\approx \left(\frac{-2}{\alpha_k+\beta_k}\right)\sum_{\mathrm{ij}} \mathrm{a}_{\mathrm{k.ij}}\alpha_i\beta_j + o(t) \\ &= -\sum_{ij}\widehat{a}_{k.ij}\alpha_i\beta_j + o(t), \quad \widehat{a}_{k.ij} = \frac{2a_{k.ij}}{\alpha_k+\beta_k}\end{aligned}$$

Risk seeking For risk seeking (rs), the sign of the Arrow-Pratt operator changes according to a spin vector [Wardle, 2008, pp. 16-17]. So, $\widehat{a}_{k.ij} \to \widehat{a}_{k.ji}$, and $\alpha_i\beta_j$ remains the same. Define

$$\theta_{ji}\alpha_j\beta_i = \alpha_j^2 + \beta_i^2 \text{ and } a_{k.ji} = (2+\theta_{ji})_k \text{ such that}$$

$$(A_{rs}\mathbf{u})_{\mathrm{k}} = (\boldsymbol{\beta\alpha})_k = \sum_{\mathrm{ij}}\widehat{\mathrm{a}}_{k.ji}\alpha_i\beta_j + o(t)$$

Risk commutator and structure constant I

Subtract the risk seeking generator from the risk aversion generator (on the same manifold) to get the k-th element of the Lie product vector:

$$
\begin{aligned}
(A_{ra}\mathbf{u})_k - (\mathrm{A_{rs}}\mathbf{u})_\mathrm{k} &= (\boldsymbol{\alpha\beta})_k - (\boldsymbol{\beta\alpha})_k \\
&= -\sum_{\mathrm{ij}} \widehat{\mathrm{a}}_{\mathrm{k.ji}}\alpha_i\beta_j + \mathrm{o}(\mathrm{t}) - \sum_{\mathrm{ij}} \widehat{\mathrm{a}}_{\mathrm{k.ij}}\alpha_i\beta_j + o(t) \\
\Rightarrow ((A_{ra} - A_{rs})\,\mathbf{u})_k &= -\sum_{i,j} (\hat{a}_{k.ij} + \hat{a}_{k.ji})\,\alpha_i\beta_j + o(t) \\
\Rightarrow ((A_{ra} - A_{rs})\,\mathbf{u})_k &\to \sum_{i,j} c_{k.ij}\alpha_i\beta_j
\end{aligned}
$$

where the quantity

$$
c_{k.ij} = -\left(\hat{a}_{k.ij} + \hat{a}_{k.ji}\right)
$$

is the structure constant for the risk operations on our topological group G.

Risk commutator and structure constant II

Definition (Commutator)

Let $\mathbf{x},\ \mathbf{y} \in G$. The *commutator* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y}$. The commutator is the element that induces commutation between $\mathbf{x}$ and $\mathbf{y}$ so that

$$\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y})$$

□

Definition (Structure constant or coupling constant)

The structure constant $c_{k.ij}$ characterizes the strength of the interaction between risk averse and risk seeking behavior. □

Theorem (Infinitesimal vector of commutator curve)

[Guggenheimer, 1977, pg. 106]
$[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ *is the infinitesimal vector of the commutator curve* $(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y})(t^2)$. □

Risk torsion

The quantities

$$\widehat{a}_{k.ij} = \left(\frac{2}{\alpha_k + \beta_k}\right) a_{k.ij}$$

has the following interpretation.

- α_k, β_k are the k-th element of the tangent vector $\dot{\mathbf{x}}(t)$ and $\dot{\mathbf{y}}(t)$ and $2a_{k.ij}$ is the k-th coefficient of the second order terms which reflect the rate of spin of the tangent vectors
- Thus, $\widehat{a}_{k.ij}$ is a *torsion type* constant. It reflects the rate at which agents "flip" between risk aversion and risk seeking in decision making. It is, in effect, *risk torsion*. Much like [Pratt, 1964, pg. 127] who distinguished his risk measure from curvature, we distinguish "risk torsion" from the torsion in [Wardle, 2008, p.19].

Coupling or risk attitudes

Lemma (Coupling risk aversion and risk seeking torsion)

The structure constant $c_{k.ij} = -\left(\widehat{a}_{k.ij} + \widehat{a}_{k.ji}\right)$ *associated with risk operations reflects the coupling between risk aversion and risk seeking torsion behavior in decision making.* □

Prudence

Definition (Prudence)

[Sandmo, 1970, pg. 353] A subject is prudent if in the face of income risk [s]he engages in *precautionary savings* as a buffer against future consumption. □

[Sandmo, 1970, pg. 359] condition for prudence rests on the relationship:

$$\frac{\partial}{\partial C_2}\left\{\frac{\frac{\partial^2 U}{\partial C_1 \partial C_2} - (1+r)\frac{\partial^2 U}{\partial C_2^2}}{\frac{\partial U}{\partial C_2}}\right\} < 0$$

where U represents a continuous preference ordering over present (C_1) and future consumption (C_2). The above implies the existence of U'''. In fact, [Sandmo, 1970, pg. 354, eq. 2] and [Kimball, 1990, pg. 60, eq. 9] imply that for a utility function $U \in C^3(X)$ prudence is defined by some operation A_{pa} (the *pa* is for "prudence risk aversion") such that

$$A_{pa}U = -\frac{U'''}{U''}$$

which, in the context of our Arrow-Pratt operator A_{ra} is a risk operation

$$A_{pa}U = A_{ra}U''$$

Prudence operations

Lemma

For some measure μ on X consider the integral operator for U in the domain of $\Im$

$$\Im(\mu) = (\Im U)(x) = \int_X U(x)\mu(dx), \qquad \textit{so that}$$

$$U = (\Im \circ \Im)U'' \Rightarrow (A_{pa} \circ \Im \circ \Im)U'' = A_{pa} \circ (\Im \circ \Im)U'' = A_{pa}U$$

$$\Rightarrow A_{ra} = (A_{pa} \circ \Im \circ \Im)$$

by virtue of the prudence operator A_{pa} for risk aversion.

Remark

We note that $\Im$ could be any one of several functional integration operators characterized by a μ-measure in the literature on decision making under risk and uncertainty. For example, $\Im$ includes but is not limited to [Von Neumann and Morgenstern, 1953](VNM utility functional for expected utility theory (EUT)); [Gilboa and Schmeidler, 1989](maximin expected utility (MEU)); [Klibanoff et al., 2005] (smooth ambiguity); [Maccheroni et al., 2006] (variational model of that captures ambiguity);[Chateauneuf and Faro, 2009] (operator representation of confidence preferences) or [Machina, 1982](local utility functional).

Coupling risk averse and risk seeking prudence actions

Let $\ominus$ be the coupling action for risk averse and risk seeking prudence operations. Thus we can rewrite the Lie product relationship above as

$$((A_{ra} - A_{rs})\,\mathbf{u})_k = ([(A_{pa} \ominus A_{ps}) \circ \mathfrak{I} \circ \mathfrak{I}]\,\mathbf{u})_k \to \sum_{i,j} c_{k.ij}\alpha_i\beta_j$$

We summarize the foregoing with the following

Theorem (Prudence risk torsion operator)

Let D be a differential operator, $A_{ra} = -D\ln D$ be Arrow-Pratt risk aversion operator, and $A_{rs} = -A_{ra}$ be the corresponding risk seeking operator. Furthermore, let $\mathfrak{I}$ be an integral operator. Define the prudence operation for risk aversion by $(A_{pa}U) = (A_{pa} \ominus A_{ps}) \circ \mathfrak{I} \circ \mathfrak{I}U''$, assuming that the expressed functions are in the domains of the respective operators. Then the prudence risk torsion operator is given by

$$(A_{ra} - A_{rs}) = [(A_{pa} \ominus A_{ps}) \circ \mathfrak{I} \circ \mathfrak{I}]$$

□

Risk operator representation

It is axiomatic that the *risk torsion operator* $A = A_{ra} - A_{rs}$ has trace $tr(A) = 0$, so it has positive and negative eigenvalues. Thus, it belongs to the *quantum group* $SU(n)$ which admits trace A=0.

Theorem (Lie algebra of risk operators)

Let G be a compact group on a differentiable manifold X of choice vectors in R^n, and $\mathbf{u} : G \times G \to G$ be a mapping of a compact group onto itself. Let $\mathbf{u}$ be a $C_0^2(X)$ vector valued von Neuman-Morgenstern utility function defined on X, and $\mathbf{x}(t), \mathbf{y}(t)$ be choice vectors in $G \subset X$. Define risk operators $A_{\{\cdot\}}$ such that for risk aversion $A_{ra} = -DlnD$ (risk seeking $A_{rs} = DlnD$) on the class of functions $u \in C_0^2(X) \cap \mathfrak{D}(A)$ where $\mathfrak{D}(A)$ is the domain of A. Then the Lie algebra $\mathcal{L}(G)$ for the risk associated to u is the special linear group SL_n of skew symmetric matrices. □

Theorem (Risk torsion quantum group)

Let $\mathbf{u}$ be a $C_0^2(X)$ vector valued von Neuman-Morgenstern utility function defined on X, and $A = A_{ra} - A_{rs}$ be a risk torsion operator. Then A has representation in the quantum group $SU(n)$. □

Applications to Behavioural Economics–Loss Aversion

Loss aversion gauge

- [Wakker and Tversky, 1993], [Tversky and Kahneman, 1992, p. 209] and [Charles-Cadogan, 2016] establish a relationship between risk aversion and risk seeking. Those papers imply that the Arrow-Pratt risk operators for risk aversion and risk seeking are not commutative.

Lemma

There exist a risk operator relationship of type $A_{ra} = -\lambda A_{rs}$ where λ is a loss aversion index and A_{ra}, A_{rs} are risk aversion and risk seeking operators. When $\lambda = 1$ we have loss neutrality. □

By virtue of the infinitesimal generator construct, $\theta_{ij}\alpha_i\beta_j = (\alpha_i^2 + \beta_j^2)$. Furthermore

$$\widehat{a}_{k.ji} \leftrightarrow \lambda \widehat{a}_{k.ij}$$

$$c_{k.ij} = -(\widehat{a}_{k.ij} + \lambda \widehat{a}_{k.ji}) = 0 \Rightarrow \theta_{ij} + \lambda\theta_{ji} + 2(1+\lambda) = 0$$

$$\Rightarrow \alpha_j^2 + \beta_i^2 = \frac{2\alpha_i\beta_j(1+\lambda) + r^2}{\lambda}, \;\; \lambda \neq 0$$

By virtue of skew symmetry we showed that $\alpha_i\beta_j = -\alpha_j\beta_i$ for real values α and β. In which case we let

$$\alpha_j^2 + \beta_i^2 = R^2$$

$$\alpha_i^2 + \beta_j^2 = r^2$$

The Loss aversion index as a gauge transformation

Motivated by [Köbberling and Wakker, 2005, pg. 125], we propose the following

Definition (Loss aversion gauge)

Loss aversion is a psychological gauge transformation which governs the rate of exchange between risk seeking and risk aversion. □

The rationale for the proposed definition is as follows.

- The structure constant, and other parameters were defined above. So the commutator depends on λ. In this case λ is a *gauge transformation* because it has no effect on the commutativity of the underlying vectors. That is, $c_{k.ij} = 0$ is invariant to inclusion of λ.
- Nonetheless, λ tells us how near or far the risk operators are from being "symmetric" or being commutative. For example, if $\lambda = 1$, then $A_{ra} = -A_{rs}$ and we are in a world of expected utility theory (EUT). When $\lambda > 1$ we are in a world of distorted EUT or non-EUT

Theorem (Nonparametric estimate for loss aversion index under uncertainty)

Let U_{α}^{MTK} be the nbd of a reference point in MTK topology. Let $\mathbf{D}$ be a unit disk, and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be vectors in a real valued Hardy space $\mathfrak{D} \subset H^p(\mathbf{D}) \cap U_{\alpha}^{MTK}$ such that $\alpha_i^2 + \beta_j^2 = r^2 > 0$, $0 < r < 1$; $1 \leq i, j \leq n$. The loss aversion index estimator for payoffs $\boldsymbol{\alpha}, \boldsymbol{\beta}$ when probabilities are not known is given by

$$\lambda_{ij} = \frac{r^2 + 2\alpha_i\beta_j}{R^2 - 2\alpha_i\beta_j}, \quad \lambda > 0, \quad 2\sup_{i,j}\{\alpha_i\beta_j\} \leq R^2$$

$R^2 = \mathtt{ceil}(2\sup_{i,j}\{\alpha_i\beta_j\})$ is a free variable on $\mathbb{R}^+$. If $0 < r < R < 1$, then the numerator and denominator of the loss aversion index λ exist in a real valued Hardy space $H^p(\mathbf{D})$. □

Remark

In a sense, $\alpha_i\beta_j$ is a covariance factor for choice between α_i and β_j. The set $\mathfrak{D} \subset H^p(\mathbf{D}) \cap U_{\alpha}^{MTK}$ which requires $2\alpha_i\beta_j \leq 1$ is too small to accommodate loss aversion index λ_{ij} as shown in applications below.

Local and global loss aversion

Theorem (Local and global loss aversion)

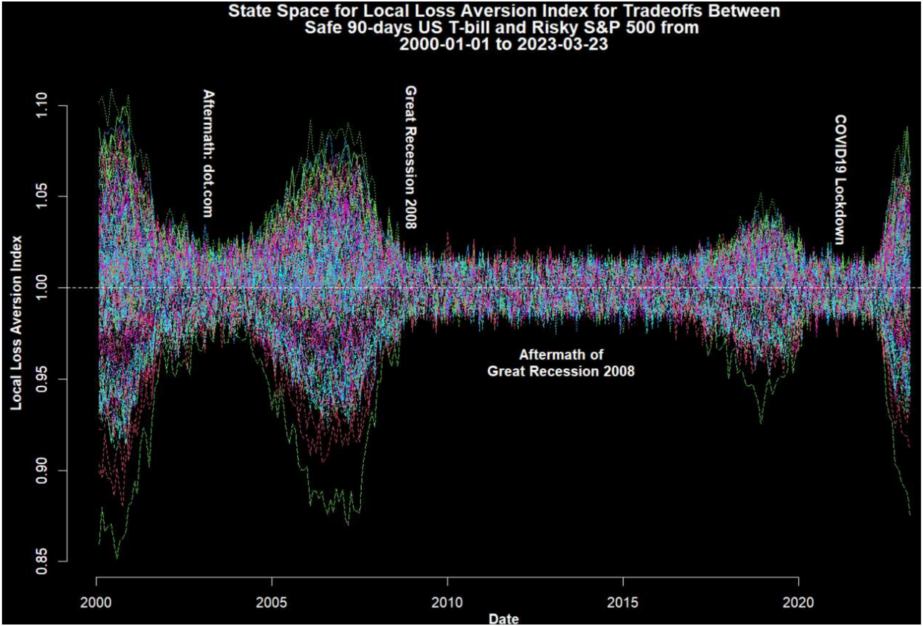
$\lambda_{LLA_{i,j}} = \lambda_{ij}$ is a local loss aversion (LLA) index, and $A_L = [\lambda_{ij}]_{1 \leq i,j \leq n}$ is matrix operator such that the global loss aversion (GLA) index is given by the Frobenius norm $\lambda_{GLA} = \|A_L\|_F$.

Remark

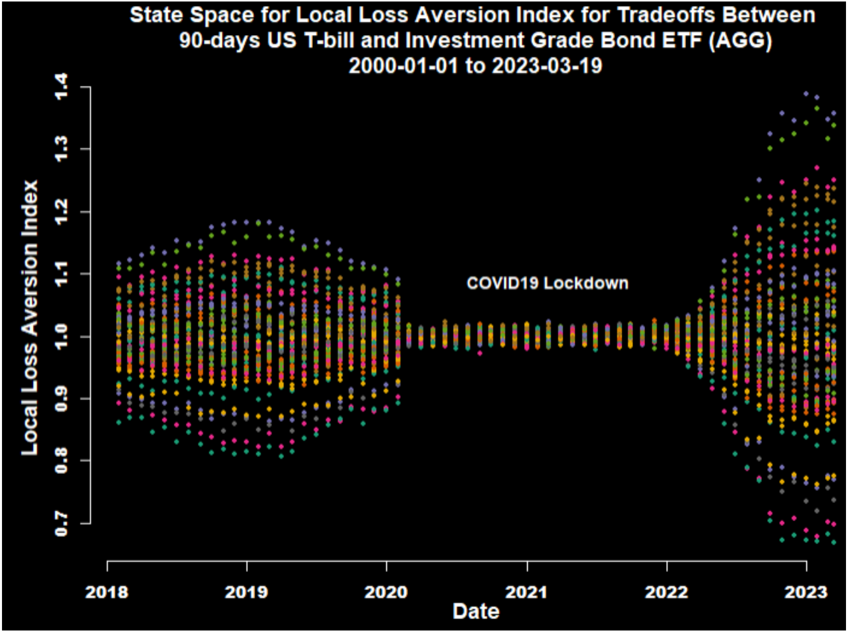
In this set up, A_L is not symmetric. Also, $\lambda_{ij} = \infty$ if $2\alpha_i \beta_j = 1$ is admissible. See e.g., [Charles-Cadogan, 2018] who proved that the loss aversion index admits a half-Cauchy distribution. We chose Frobenius norm because of its distance metric feature. However, the choice of matrix norm is open since it is known that all norms in $\mathbb{R}^{n \times n}$ are functionally equivalent. Refer to [Golub and Van Loan, 2013, pp. 72-73] for examples of matrix norms.

Data visualization implied by representation theory of risk operators

Sample paths of implied loss aversion in stock markets

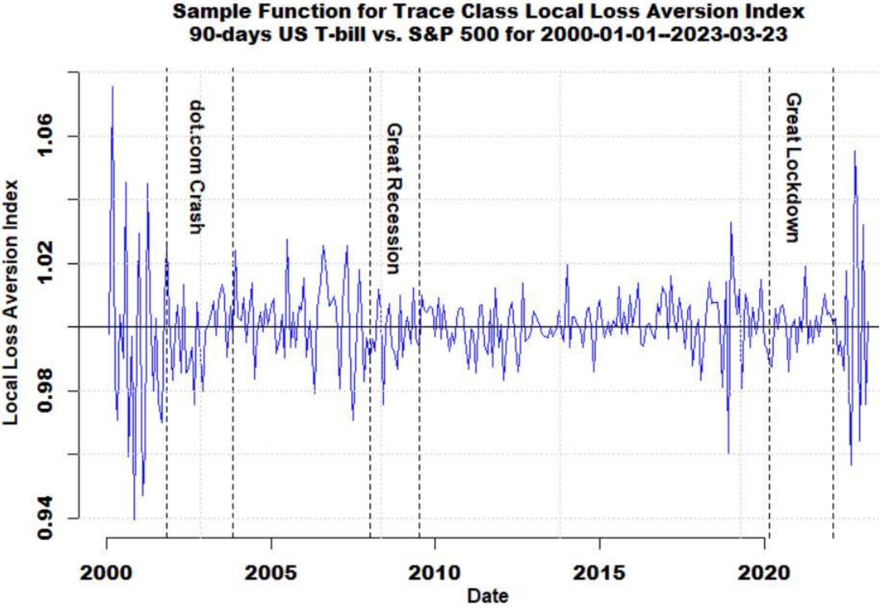


Sample paths of implied loss aversion in bond markets

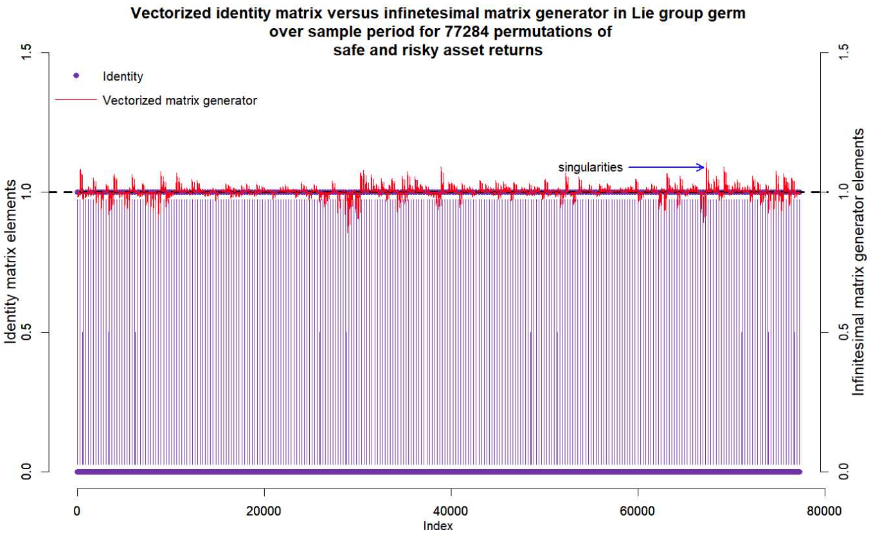


The plot depicts the 5-years cut between 2018 and 2023 for the data beginning in 2000.

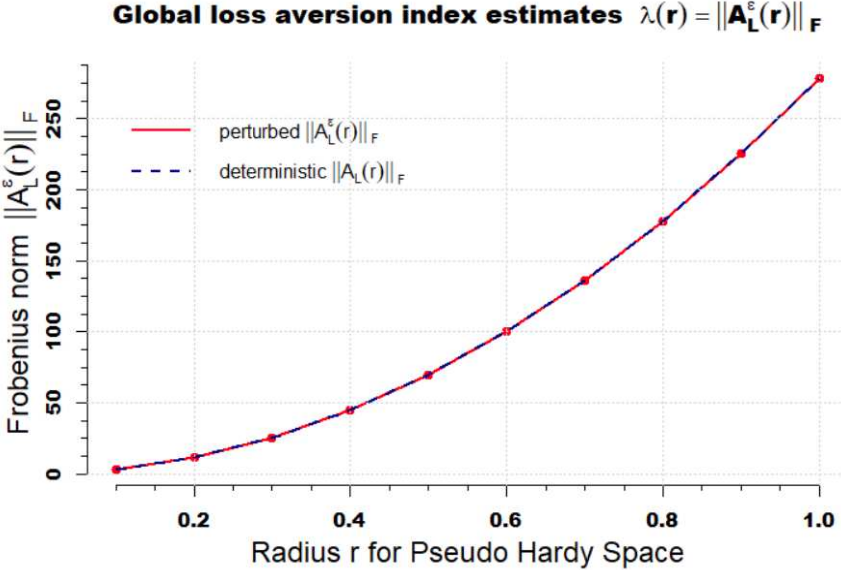
The trace class sample paths of implied loss aversion in financial markets



Singularities of implied loss aversion in financial markets



Global implied loss aversion in financial markets



Conceptually, we impose 10 "concentric disks" in steps of 0.1 inside a pseudo real valued Hardy space, and estimate the implied loss aversion index supported on each disk.

Asset Pricing

Table: Fama-French Quintile Sort Value Weighted Operating Profit

	Dependent variable:					
	Qnt2		Qnt3		Qnt4	
	Qnt2rm	Qnt2rl	Qnt3rm	Qnt3rl	Qnt4rm	Qnt4rl
	(1)	(2)	(3)	(4)	(5)	(6)
Constant	0.13^{*}	-145.68^{***}	0.18^{**}	-147.53^{***}	0.20^{***}	-147.70^{***}
	(0.07)	(16.37)	(0.07)	(15.98)	(0.06)	(14.71)
MktRP	1.02^{***}		1.01^{***}		0.95^{***}	
	(0.02)		(0.02)		(0.01)	
LLAindex		146.40^{***}		148.28^{***}		148.43^{***}
		(16.33)		(15.95)		(14.67)
SMB	0.02	0.30^{***}	-0.06^{**}	0.22^{**}	-0.05^{**}	0.22^{***}
	(0.03)	(0.09)	(0.03)	(0.09)	(0.02)	(0.08)
HML	0.21^{***}	0.58^{***}	0.22^{***}	0.58^{***}	0.06^{***}	0.40^{***}
	(0.03)	(0.09)	(0.03)	(0.09)	(0.02)	(0.08)
RMW	-0.20^{***}	-0.53^{***}	-0.04	-0.36^{***}	0.16^{***}	-0.13
	(0.03)	(0.10)	(0.03)	(0.10)	(0.03)	(0.09)
CMA	0.07^{*}	-0.41^{***}	-0.04	-0.51^{***}	-0.12^{***}	-0.55^{***}
	(0.04)	(0.14)	(0.04)	(0.14)	(0.04)	(0.12)
Observations	276	276	276	276	276	276
R^2	0.95	0.45	0.95	0.41	0.95	0.40
Adjusted R^2	0.95	0.44	0.95	0.40	0.95	0.39
Residual Std. Error	1.14	3.85	1.10	3.76	0.96	3.46
F Statistic	$1,050.47^{***}$	43.35^{***}	$1,009.51^{***}$	37.82^{***}	$1,109.40^{***}$	36.03^{***}
Note:						*p<0.1; **p<0.05; ***p<0.01

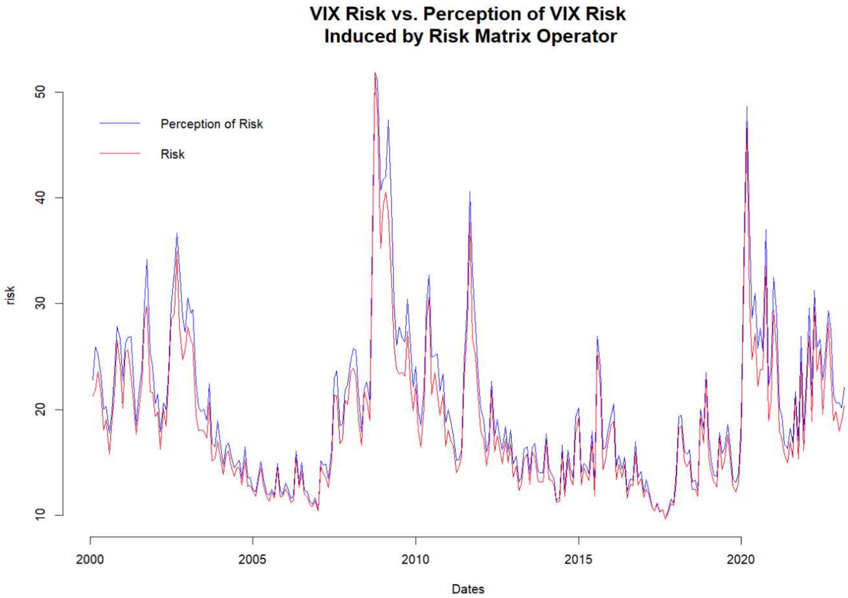
Data taken from Ken French website in (%). On average, the trace class *LLAindex* factor seems to have more statistical test power than the market risk premium factor *MktRP*. This is an artifact of the trace being based on the invariant core of risk operations. However, *MktRP* has more explanatory power. In a separate regression we find $\widehat{MktRP} = 162 \times (-1 + LLAindex)$, $p < 0.001$. This implies that the predicted market risk premium is 0 for loss neutrality (i.e., when $LLAindex = 1$). So, our model is internally consistent.

Table: Fama-French Book-to-Market 30-40-30 (growth, mid, value) Value Weighted Operating Profit

	Dependent variable:					
	Lo30		Med40		Hi30	
	Lo30rm	Lo30rl	Med40rm	Med40rl	Hi30rm	Hi30rl
	(1)	(2)	(3)	(4)	(5)	(6)
Constant	0.07	−159.68***	0.16***	−148.26***	0.16***	−139.12***
	(0.05)	(16.17)	(0.05)	(15.42)	(0.04)	(14.98)
MktRP	1.06***		1.00***		0.97***	
	(0.01)		(0.01)		(0.01)	
LLAindex		160.34***		148.98***		139.84***
		(16.13)		(15.39)		(14.95)
SMB	0.10***	0.39***	−0.03*	0.25***	−0.07***	0.20**
	(0.02)	(0.09)	(0.02)	(0.08)	(0.01)	(0.08)
HML	0.16***	0.53***	0.16***	0.51***	−0.11***	0.23***
	(0.02)	(0.09)	(0.02)	(0.09)	(0.01)	(0.08)
RMW	−0.59***	−0.92***	−0.01	−0.32***	0.29***	−0.02
	(0.02)	(0.10)	(0.02)	(0.10)	(0.02)	(0.09)
CMA	−0.15***	−0.63***	−0.05	−0.51***	0.08***	−0.38***
	(0.03)	(0.14)	(0.03)	(0.13)	(0.02)	(0.13)
Observations	276	276	276	276	276	276
R^2	0.98	0.60	0.97	0.42	0.98	0.32
Adjusted R^2	0.98	0.59	0.97	0.41	0.98	0.31
Residual Std. Error	0.80	3.80	0.80	3.63	0.56	3.52
F Statistic	3,009.34***	81.34***	1,882.28***	39.55***	3,139.04***	25.58***
Note:						*p<0.1; **p<0.05; ***p<0.01

The *LLAindex* factor seems to have more power for growth and middle 40% value weighted operating profit portfolios. However, it has less power for value portfolios. Nonetheless, the Fama-French factors retain superior explanatory power.

Perceptions of risk in stock markets



Perceptions of risk were computed with the reduced row-echelon form of the underlying infinitesimal matrix operator.

Thank you

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