From stochastic decision making in non-conflict situations to quantum computing and topological quantum computing

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1. Need for stochastic decision making in conflict situations

- In many practical situations, we need to make a decision.
- A naive commonsense understanding is that the decision needs to be deterministic.
- If a person flips a coin to decide where to have lunch, this usually means that this person does not care.
- In serious situations, we are supposed to think and make a thought-of decision.
- In their seminal book on game theory, von Neumann and Morgenstern give a good example of such an attitude.
- If an admiral explains that he selected a route for the ships by flipping a coin, this admiral will be dismissed right away.

2. Need for stochastic decision making in conflict situations (cont-d)

- However, in conflict situations, it is often advantageous to make stochastic decisions.
- For example, in rock-paper-scissor game, a person applying a deterministic strategy will lose.
- The optimal strategy is to select each of the three possible moves with equal probability 1/3.

3. But do we need stochastic decision making in non-conflict situations? Looks like no

- But what if we have a non-conflict situation: a person selects between several options.
- According to decision theory, decisions by rational person can be described as follows.
- To each alternative i, we can assign a numerical value u_i known as its *utility*.
- We always select the alternative with the largest possible value of utility.
- The utility of a "lottery" in which we get alternative i with probability p_i is equal to

$$u = p_1 \cdot u_1 + \ldots + p_n \cdot u_n, \text{ where } p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1.$$

4. But do we need stochastic decision making in non-conflict situations? Looks like no (cont-d)

- One can easily see that the largest value of u is attained when we take the option with largest utility – with probability 1.
- If we have several options with equal utility, we can choose each with some probability.
- However, we do not gain anything this way.
- And in real life, exact utility values are rare.
- So, it looks like we do not need stochastic decision making in non-conflict situations.

5. But do we need stochastic decision making in non-conflict situations? Actually yes

- The above argument was based on classical physics, where uncertainty is described by probabilities p_i.
- In quantum physics, uncertainty is, in general, described by a symmetric *density matrix* $\rho_{i,j}$.
- We can have different such states with different probabilities p_i, resulting in
$$\rho_{i,j} = p_1 \cdot \rho^{(1)}_{i,j} + \ldots + p_n \cdot \rho^{(n)}_{i,j}.$$
- We know that the utility of a probabilistic combination is equal to the probabilistic combination of utilities.
- This implies that, in general, the utility is equal to $u = \sum\limits_{i,j} \rho_{i,j} \cdot u_{i,j}$ for some symmetric matrix $u_{i,j}$.
- Here, the constraint is that $\sum\limits_{i} \rho_{i,i} = 1$ and that the matrix $\rho_{i,j}$ is non-strictly positive definite.

6. But do we need stochastic decision making in non-conflict situations? Actually yes (cont-d)

- In this case, the largest value of u is not necessarily attained for a deterministic decision.
- One can show that its is attained when $\rho_{i,j} = a_i \cdot a_j$ for the eigenvector a_i of $u_{i,j}$ corresponding to the largest eigenvalue.
- So, if we take quantum uncertainty into account, even in non-conflict situations we should make stochastic decisions.

7. Natural next question: how to find the optimal stochastic decision

- From the practical viewpoint, the important issue is how to compute the optimal decision.
- Many decision problems are difficult, they need a lot of computations.
- The need to select probabilities, and not just one of the alternatives, makes it even more difficult.
- Interestingly, as we will show, this need leads us again to quantum effects – namely, to quantum computing.

8. Need for faster computers

- Modern computers are extremely fast.
- However, there are still many important practical problems for which the current computer speed is not sufficient.
- One of such problems is the problem of tornado prediction.
- In many areas of the US, destructive tornados appear year after year, bringing lot of destructions and even deaths.
- Once a tornado is sighted, a warning is issued.
- In principle, people have access to shelters.
- However, during the tornado season, warnings are issued practically every day, and people cannot spend all their lives in shelters.
- Besides, for each town, the vast majority of tornados do not enter this town's area.
- As a result, people ignore the warnings, and once in a while a disaster happens.

9. Need for faster computers (cont-d)

- The only way to prevent such disasters is to be able to reasonably accurately predict in what direction a tornado will move.
- This way, warning will be issued only to people in danger, and others will be able to continue their normal activities.
- In principle, such predictions are possible.
- After all, tornado is an atmospheric effect just like storms and hurricanes, and we know how to predict weather.
- In particular, we know how to predict in what direction storms and hurricanes will move.
- By spending an hour or so on a supercomputer, we can get a very good understanding of where a storm will move.
- Similarly, by spending an hour or so on a high performance computer, we can estimate in what direction a tornado will turn.

10. Need for faster computers (cont-d)

- The problem is that tornados are smaller in size and thus, their dynamics is faster.
- Whatever changes occur to a storm in a day takes 15 minutes for a tornado.
- Thus, the fact that we can predict the tornado dynamics by spending an hour on a supercomputer is useless.
- By the time we finish computations, the tornado has already changed directions four times.
- Thus, we need to make computers much faster.
- There are many other practical problems in which the same need appears.
- To speed up computations, we need to make computer components smaller.

11. Need for faster computers (cont-d)

- A fundamental limit to computation speed is the fact that all communication speeds are limited by the speed of light.
- Indeed, in a usual laptop of 30 cm size, it takes 1 nanosecond for light to travel from one side to another.
- During this time, the simplest 4 GHz processor already performs 4 operations.
- To make computations faster, we need to make computers much smaller – and thus, we need to make all the components much smaller.

12. Need for quantum computing

- When we decrease the size of computer components, we get sizes comparable to sizes of molecules and atoms.
- At this level, we cannot rely on the usual Newton's physics.
- We need to take into account that for objects of this small size, quantum effects are essential.
- Thus, we arrive at the need for computing that takes quantum effects into account – which is known as *quantum computing*.

13. Beyond the current quantum computing, to topological quantum computing

- What if we will need even faster computers?
- In this case, we will need to get to sizes which are much much smaller than the sizes of molecules and atoms.
- At certain sizes, according to modern physics, quantum fluctuations become so large that we can no longer talk about metric.
- All if left is topology – and causal order.
- Topological level is difficult to describe but appropriate for computations.
- From the mathematical viewpoint, physics at this level is much more difficult to describe.

14. Beyond the current quantum computing, to topological quantum computing (cont-d)

- Indeed, in the absence of metric, all topologically equivalent spaces are indistinguishable.
- There are no longer continuous quantities whose changes are described by usual differential equations.
- The only characteristics that distinguish different topological spaces are discrete characteristics – such as homotopy and homology groups.
- Physics does not have much experience with dynamics of such discrete structures.

15. Beyond the current quantum computing, to topological quantum computing (cont-d)

- Interestingly, from the computational viewpoint:
  - this discrete character is exactly what the doctor ordered,
  - since all computations are, by definition, discrete.
- When we get to this level, we will not face the usual challenge of simulating discrete structures on a continuous domain.
- The domain will be discrete by definition.

16. This is related to Sakharov's ideas

- This discreteness is related to the old idea of Andrei Sakharov that:
  - all physically observable discrete quantities (like electric and other charges)
  - are actually discrete characteristics of the underlying micro-level topological structure.
- This explains their discrete character.
- If this idea is true, then topological quantum computers may be easier to design than it may seem.
- Indeed, the mysterious topological characteristics may be something like charges that we observe (and handle) anyway.

17. Conclusion

- What is the relation between decision making and quantum physics?
- In pre-quantum physics, in non-conflict situations, deterministic decisions are optimal.
- However, if we take quantum effects into account, optimal decisions become stochastic.
- This stochastic character makes it more difficult to compute optimal decisions.
- In many practical situations, we need to make a decision fast.
- How can we speed up computations?
- It turns out that the need for this speed-up naturally leads us to the need to utilize quantum effects – i.e., to quantum computing.
- And in the distant future, we will need to use topological quantum computing.

18. References

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20. Traditional Decision Theory: Reminder

- To make a decision, we must:
  - find out the user's preference, and
  - help the user select an alternative which is the best – according to these preferences.
- Traditional approach is based on an assumption that for each two alternatives A' and A'', a user can tell:
  - whether the first alternative is better for him/her; we will denote this by $A'' < A'$;
  - or the second alternative is better; we will denote this by $A' < A''$;
  - or the two given alternatives are of equal value to the user; we will denote this by $A' = A''$.

21. The Notion of Utility

- Under the above assumption, we can form a natural numerical scale for describing preferences.
- Let us select a very bad alternative A_0 and a very good alternative A_1.
- Then, most other alternatives are better than A_0 but worse than A_1.
- For every prob. $p \in [0,1]$, we can form a lottery $L(p)$ in which we get A_1 w/prob. p and A_0 w/prob. $1-p$.
- When $p = 0$, this lottery simply coincides with the alternative A_0: $L(0) = A_0$.
- The larger the probability p of the positive outcome increases, the better the result:

$$p' < p'' \text{ implies } L(p') < L(p'').$$

22. The Notion of Utility (cont-d)

- Finally, for $p = 1$, the lottery coincides with the alternative A_1: $L(1) = A_1$.
- Thus, we have a continuous scale of alternatives $L(p)$ that monotonically goes from $L(0) = A_0$ to $L(1) = A_1$.
- Due to monotonicity, when p increases, we first have $L(p) < A$, then we have $L(p) > A$.
- The threshold value is called the *utility* of the alternative A:

$$u(A) \stackrel{\text{def}}{=} \sup\{p : L(p) < A\} = \inf\{p : L(p) > A\}.$$

- Then, for every $\varepsilon > 0$, we have

$$L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon).$$

- We will describe such (almost) equivalence by $\equiv$, i.e., we will write that $A \equiv L(u(A))$.

23. Fast Iterative Process for Determining $u(A)$

- *Initially:* we know the values $\underline{u} = 0$ and $\overline{u} = 1$ such that $A \equiv L(u(A))$ for some $u(A) \in [\underline{u}, \overline{u}]$.
- *What we do:* we compute the midpoint u_{mid} of the interval $[\underline{u}, \overline{u}]$ and compare A with $L(u_{\text{mid}})$.
- *Possibilities:* $A \le L(u_{\text{mid}})$ and $L(u_{\text{mid}}) \le A$.
- *Case 1:* if $A \le L(u_{\text{mid}})$, then $u(A) \le u_{\text{mid}}$, so
$$u \in [\underline{u}, u_{\text{mid}}].$$
- *Case 2:* if $L(u_{\text{mid}}) \le A$, then $u_{\text{mid}} \le u(A)$, so
$$u \in [u_{\text{mid}}, \overline{u}].$$
- After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half.
- After k iterations, we get an interval of width 2^{-k} which contains $u(A)$ – i.e., we get $u(A)$ w/accuracy 2^{-k}.

24. How to Make a Decision Based on Utility Values

- Suppose that we have found the utilities $u(A')$, $u(A'')$, ..., of the alternatives A', A'', ...
- Which of these alternatives should we choose?
- By definition of utility, we have:
  - $A \equiv L(u(A))$ for every alternative A, and
  - $L(p') < L(p'')$ if and only if $p' < p''$.
- We can thus conclude that A' is preferable to A'' if and only if $u(A') > u(A'')$.
- In other words, we should always select an alternative with the largest possible value of utility.
- Interval techniques can help in finding the optimizing decision.

25. How to Estimate Utility of an Action

- For each action, we usually know possible outcomes $S_1, \ldots, S_n$.
- We can often estimate the prob. $p_1, \ldots, p_n$ of these outcomes.
- By definition of utility, each situation S_i is equiv. to a lottery $L(u(S_i))$ in which we get:
  - A_1 with probability $u(S_i)$ and
  - A_0 with the remaining probability $1 - u(S_i)$.
- Thus, the action is equivalent to a complex lottery in which:
  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.

26. How to Estimate Utility of an Action (cont-d)

- *Reminder:*
  - first, we select one of the situations S_i with probability p_i: $P(S_i) = p_i$;
  - then, depending on S_i, we get A_1 with probability $P(A_1 \mid S_i) = u(S_i)$ and A_0 w/probability $1 - u(S_i)$.
- The prob. of getting A_1 in this complex lottery is:

$$P(A_1) = \sum_{i=1}^{n} P(A_1 \mid S_i) \cdot P(S_i) = \sum_{i=1}^{n} u(S_i) \cdot p_i.$$

- In the complex lottery, we get:
  - A_1 with prob. $u = \sum\limits_{i=1}^{n} p_i \cdot u(S_i)$, and
  - A_0 w/prob. $1 - u$.
- So, we should select the action with the largest value of expected utility $u = \sum p_i \cdot u(S_i)$.