The Topology of Quantum Theory and Social Choice

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The Topology of Quantum Theory and Social Choice

- Quantum Theory emerged from the axioms created by Born and von Neumann nearly 100 years ago and is now the most succesful scientific theory of all time.
- The first part of this presentation defines the concept of **unicity**, and shows why it is critical in quantum theory. Topology is key to unicity and therefore to quantum theory.
- The second part of the presentation establishes the **connection between quantum theory and social choice**, an unexpected and important link between the physical and the social sciences. Topology is also critical in this link betweem physical and social sciences
- The presentation as a whole is based on the definitions and theorems in "The Topology of Quantum Theory and Social Choice" **Quantum Reports** 2022,4, 201-229 https:doi.org/103390/4020014 LaTeX.

Part I: Unicity and the Violation of Unicity

A fundamental difference between classic physics and quantum physics is that classic physics studies mathematical models of a single experiment, or of subexperiments of a single experiment. Quantum physics is more ambitious: it seeks mathematical models for the class of all possible experiments that can be performed on a physical system. The single experiment that comprises all experiments in classic physics is called the **universe**. **Unicity** in classic physics is the requirement that all events are included in one single sample space. It is usually expressed as the assumption in classic physics that there is a **single universe**. Such an assumption does not exist in quantum physics: the Unicity Postulate is not required in Quantum Physics. In particular the experiments performed on a physical system may or not be consistent with each other, and the concept of **interference** which is unique to quantum physics arises precisely with respect to that possibility.

Part II: The Violation of Unicity creates Singularities that Explain Quantum Theory Puzzles

Based on von Neumann's axioms we identify a class of singularities that separates classic from quantum physics and explains many quantum theory puzzles. These singularities are topological objects that provide new experimental insights. The key is the **topology of spaces of quantum events and of frameworks. In quantum physics these are topologically complex, while in classic physics the spaces of events and of frameworks are topologically trivial.** This explains entanglement, Heisenberg uncertainty, order dependence of observations, the conjunction fallacy and other important puzzles. Somewhat surprisingly the same topological singularities explain the classic **social choice paradox** as formalized in Chichilnisky (1980).We identify necessary and sufficient conditions to resolve the quantum paradoxes: the same conditions resolve social choice paradox and the quantum paradoxes.

Part III: Singularities and the Observer: the Unicity Postulate and the Topology its absence Creates

The topological singularities appearing in quantum physics can be identified with the presence of an **Observer.**

Recall that the Unicity Postulate is the requirement that all events are included in the same, unique, sample space.

The Unicity Postulate is not required in Quantum, Physics.

While classic physics attempts to explain the universe, quantum theory shares with general relativity an emphasis on the Observer. For this reason 'quantum events" are defined as maps rather than as measurable sets of objecs as in classic physics.

Quantum events are a key concept in this article and they are identified with projection maps (see Axiom 2 below) and with the subspaces of a Hilbert Space onto which the projections map. Frameworks are orthonormal basis of coordinates of the Hilbert space and can be identified with subspaces of the space of events. When two frameworks fail to be orthonormal the corresponding experiments are said to "interfere" with each other. In classic physics things are different: there is only one experiment - the "universe" - and one single framework, so quantum interference is impossible: this is the "unicity hypothesis" that defines classic theory and is violated in quantum theory. **A key difference between classic physics and quantum physics is that quantum theory does not assume a single framework nor a single sample space.**

Von Neumann Axioms of Quantum Theory

(A1) The states of a quantum system are unit vectors in a complex Hilbert Space $\mathcal{H}$.
(A2) The Observables are self-adjoint operators in $\mathcal{H}$
(A3) The probability that an observable T has a value in a Borel set $A \subseteq \mathbf{R}$ when the system is in state ψ is $< P^T(A)\psi, \psi >$ where $P^T(\cdot)$ is the resolution of the identity (i.e. **spectral measure**) for T .
(A4) If the state at time $t = 0$ is ψ then at time t it is $\psi_t = e^{-itH/\hbar}\psi$ where H is the energy observable and $\hbar$ is Planck's constant

The main axiom is (A2):

Axiom (A2): the events (or observables) of a quantum system can be represented by self adjoint projections on a Hilbert Space.

Topological differences between classic and quantum physics

Classic Physics and Quantum Physics - simple example of the differences in spaces of events:

Lemma 1: The space of classic events $\sum$ in R^2 is the Boolean σ algebra of Borel measurable sets in R^2.This is a convex space and is therefore topologically trivial (i.e.all homotopy groups are zero). Unicity is satisfied since there is a unique sample space namely R^2. The space of classical events has a single element. In contrast the space of quantum events is the space of all unoriented lines through the origin within two dimensional space R^2 namely the two dimensional projective space P^2 which is also the Grassmanian space of 1- spaces in R^2 and can be identified with the one sphere S^1. When $n = 2$, the space of (quantum) frameworks can be identified with the space of quantum events F^2 and both can be identified with the projective space P^2 the Grassmanian of one planes within R^2.Neither the space of quantum events nor the space of (quantum) frameworks are contractible.

Topological Differences between Classic and Quantum Physics

In the following we consider spaces of finite dimensions n,where n is arbitrarily large, namely Euclidean spaces R^n,which correspond to physical systems with n degrees of freedom. Under appropriate assumptions the theory presented here can be made applicable to infinite dimensional Hilbert spaces. The finite dimensional case is useful to simplify the presentation and to show that fundamental properties of quantum theory occur even with finite dimensional real Hilbert spaces even though full generality requires infinite dimensional Hilbert spaces with complex coefficients.

Definitions

A map $\Psi : X^k \rightarrow X$ is symmetric when it does not depend on the order of its arguments. It is called the identity on the diagonal when for all $x, \Psi(x, x, ..., x) = x$.A framework in R^n is an $n-$ dimensional unordered orthonomal basis of coordinates in R^n. A framework selection is a way to select a single framework among any $k > 1$ frameworks which is both symmetric and is the identity on the diagonal.

Theorem

In a physical system with $n > 1$ degrees of freedom the space of quantum events can be identified with the union of all the Grassmanian manifolds $G(k, n)$ the space of k dimensional subspaces of R^n for all k. For $n > 1$,the manifold of all frameworks is a connected subspace of the manifold of all events in R^n.

Theorem

The space F of frameworks of R^n can be identified with the space of linear preferences P^n on R^n.

Theorem

(Social Choice, Chichilnisky, 1980) There is no continuous function $\Phi : S^n \times ... \times S^n$ *that is symmetric and satisfies unanimity. The same is true replacing* P^n *for the space of frameworks* F^n *and for the space of all linear preferences* P^n.

Theorem

(Unicity is equivalent to Social Choice) For any restricted domain of preferences X the social choice problem of existence of a continuous function $\Phi : X^n \times ... \times X^n$ that is symmetric and satisfies unanimity can be solved if and only if X is topologically trivial (i.e. contractible). The same is true for the existence of common frameworks or unicity. In other words, the existence of a common preference is equivalent to the existence of common frameworks or unicity.