Quantum-like modeling nondistributivity of human logic and violation of response replicability effect

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March 23

Ozawa, M. and Khrennikov, A.: Nondistributivity of human logic and violation of response replicability effect in cognitive psychology. J. Math. Psych. 112, 102739 (2023).

Basieva, I., Khrennikov, A., and Ozawa, M.: Quantum-like modeling in biology with open quantum systems and instruments. Biosystems 201, 104328 (2021).

The aim of this talk is to promote quantum logic as one of the basic tools of human reasoning.

We compare it with classical (Boolean) logic and highlight the role of violation of the distributivity law for conjunction and disjunction.

It is shown that nondistributivity is equivalent to incompatibility of logical variables - the impossibility to assign jointly the values to these variables.

We motivate that the use of quantum logic, concretely incompatible variables, speeds up the process of reasoning.

The natural question whether quantum logical nondistributivity can be tested experimentally arises.

We found that testing of the response replicability effect is equivalent to testing nondistributivity –

under the assumption that the mental state update generated by observation is described as orthogonal projection.

The simple test of RRE is suggested.

In contrast to the previous works in quantum-like modeling, we proceed in the state-dependent framework; in particular, distributivity, compatibility, and RRE are considered for a fixed mental state.

Basics of quantum logic

Operations of quantum logic are defined on the set of subspaces of Hilbert space H or equivalently on the set of orthogonal projectors $\mathcal{P}(H)$.

Subspaces (projectors) are interepreted as mathematical representations of propositions (events).

Let P be an orthogonal projector. Denote by L_P its image, i.e., $L_P = PH$. For a subspace L, denote by P_L the corresponding orthogonal projector.

For projector P, denote the projector onto the orthogonal complement to the subspace L_P by the symbol $\overline{P}$, i.e., $H = L_P \oplus L_{\overline{P}}$.

Negation of proposition P is represented by $\overline{P}$.

The operations of conjunction $\wedge$ and disjunction $\vee$ are defined as follows.

Let P and Q be an orthogonal projectors representing some propositions. The conjunction-proposition (event) $P \wedge Q$ is defined as the projector on intersection of subspaces L_P and L_Q, i.e., $L_{P\wedge Q} = L_P \cap L_Q$.

We remark that this operation is well defined even for noncommuting projectors, i.e., incompatible quantum observables. Moreover, it is commutative:

$$P \wedge Q = Q \wedge P \tag{1}$$

The same can be said about the operation of disjunction. Here subspace $L_{P\vee Q}$ is defined as the subspace generated by the union of subspaces L_P and L_Q, i.e., $P \vee Q$ is projector on this subspace.

This operation is also well defined for non-commuting projectors and, moreover, it is commutative:

$$P \vee Q = Q \vee P \tag{2}$$

Thus, quantum logic is commutative logic.

Typically this fact is not highlighted. Thus, in quantum reasoning noncommutativity is not present at the level of the basic operations of quantum logic, conjunction and disjunction.

Interrelation of distributivity and commutativity

Two propositions We start with the simplest form of distributivity law, for proposition P and Q and $R = \overline{Q}$, negation of Q.

We remark that, for any proposition Q, we have $Q \vee \overline{Q} = I$, identity operator.

Then, the distributivity law can be written in the form:

$$P = (P \wedge Q) \vee (P \wedge \overline{Q}), \tag{3}$$

i.e.,

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)), \tag{4}$$

in the same way

$$Q = (Q \wedge P) \vee (Q \wedge \overline{P}), \tag{5}$$

The following theorem is cornerstone of our modeling:

Theorem A. The distributivity law in the form (3), (5) or equivalently

$$I = (P \wedge Q) \vee (P \wedge \overline{Q}) + (\overline{P} \wedge Q) \vee (\overline{P} \wedge \overline{Q}) \tag{6}$$

holds if and only if the projectors commute, i.e., $[P, Q] = 0$.

We set

$$(7)\quad \mathrm{com}(\mathrm{P},\mathrm{Q}) = (\mathrm{P}\wedge\mathrm{Q})\vee(\mathrm{P}\wedge\overline{\mathrm{Q}}) + (\overline{\mathrm{P}}\wedge\mathrm{Q})\vee(\overline{\mathrm{P}}\wedge\overline{\mathrm{Q}})$$

and

$$(8)\qquad d(P,Q) = I - \mathrm{com}(\mathrm{P},\mathrm{Q}).$$

In quantum logic, quantities $\mathrm{com}(\mathrm{P},\mathrm{Q})$ and $d(P,Q)$ are the measures of distributivity and nondistributivity, respectively; $\mathrm{comp}(\mathrm{P},\mathrm{Q}) = 1$ or $d(P,Q) = 0$ in the distributive case.

By Theorem A, distributivity of propositions is equivalent their compatibility, i.e., the possibility to assign join values to them, say $P = 1, Q = 1$ or $P = 0, Q = 1$.

We remark that operator com(P, Q) is Hermitian. By axiomatics of quantum theory it represents an observable.

Thus, by Theorem A the distributivity law can be represented via a quantum observable.

Theoretically by measurement of this observable it is possible to check the distributivity for two propositions.

However, it seems to be difficult to present the concrete measurement procedure of this observable.

It reflects the similar problem with experimental checking of incompatibility.

Consider the Hermitian operator $i[P, Q]$. Theoretically by its measurement it is possible to check compatibility. However, the measurement procedure is not straightforward.

Typically the lattice of projectors $\mathcal{P}(H)$ is considered as union of Boolean algebras, representing classical sub-logic of quantum logic.

In this construction, the essence of Booleanity is commutativity of projectors.

However, commutativity of projectors has no straightforward logical interpretation.

And distributivity of the basic operation conjunction and disjunction is the basic law of (classical) logic.

Now, in the view of Theorem A, we can characterize classical logic as the domains of validity of the distributivity law.

Three propositions and their negations Consider now interrelation of commutativity and distributivity for three propositions $\{P, Q, R\}$. Per definition the triple is commutative if and only if each pair $\{P, Q\}, \{P, R\}, \{Q, P\}$ is commutative.

To couple commutativity and distributivity, we need to consider not only these propositions, but also their negations $\overline{P}, \overline{Q}, \overline{R}$ (otherwise the relation between commutativity and distributivity is not clear).

By distributivity of $\{P, Q, R\}$ we mean validity of equality

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)), \tag{9}$$

where $X, Y, Z = P, Q, R, \overline{P}, \overline{Q}, \overline{R}$.

Triple $\{P, Q, R\}$ is commutative if and only (9) holds, i.e., the lattice $\mathcal{P}(P, Q, R)$ generated by $\{P, \overline{P}, Q, \overline{Q}, R, \overline{R}\}$

is distributive, or ortholattice generated by $\{P, Q, R\}$ is a Boolean algebra.

These conditions are equivalent to equality

$$\mathbf{com(P, Q, R) = I,} \tag{10}$$

where

(11)

$$\mathbf{com(P, Q, R) = (P{\wedge}Q{\wedge}R){\vee}(P{\wedge}Q{\wedge}\overline{R}){\vee}(P{\wedge}\overline{Q}{\wedge}R){\vee}(P{\wedge}\overline{Q}{\wedge}\overline{R}){\vee}(\overline{P}{\wedge}}$$

$$\vee(\overline{\boldsymbol{P}} \wedge \boldsymbol{Q} \wedge \overline{\boldsymbol{R}}) \vee (\overline{\boldsymbol{P}} \wedge \overline{\boldsymbol{Q}} \wedge \boldsymbol{R}) \vee (\overline{\boldsymbol{P}} \wedge \overline{\boldsymbol{Q}} \wedge \overline{\boldsymbol{R}}).$$

We also introduce the measure of nondistributivity

$$d(P, Q, R) = I - \mathrm{com}(P, Q, R) \tag{12}$$

It equals zero in the distributive case.

Operator com(P, Q, R) is Hermitian and it represents a quantum observable, the distributivity observable. But, we repeat that design of the corresponding measurement procedure is not straightforward.

Classical vs. quantum reasoners Since the seminal works of Boole [], classical Boolean logic is widely used for modeling of human reasoning, including its probabilistic counterpart in the form of Bayesian inference.

Information processing systems using Boolean and quantum logic, respectively, are called classical and quantum reasoners.

We remark that reasoners need not be individuals, they can be special networks in the brain. Our aim is to show advantages of quantum reasoning.

The main advantage of quantum reasoning is proposition operating without such a rigid logical constraint as the distributivity law.

Quantum reasoner can derive conclusions which are nonderivable by classical reasoner.

Generality of decision making is not the only advantage of quantum reasoning.

By Theorem A matching the distributivity law is equivalent commutativity.

The latter has the meaning of assignment of the concrete values to all observables.

Classical reasoner has to process all such possibilities, for commuting propositions $P_1, ..., P_N$, these are 2^N values. This number is exponentially increasing.

By Theorem A, quantum reasoner processes information without assigning the values to propositions.

He "understands" that it cannot be done consistently.

This sort of reasoning save a lot of computation resources and speeds up information processing.

Quantum and classical reasoning can be indirectly linked to Kahneman's "fast and slow thinking".

Quantum Platonism Reasoning based on quantum logic is state-independent.

It reflects intrinsic logic of interrelation between propositions.

We can compare such viewpoint on propositions with Platonism as universals existing independently of particulars, in our case systems' states.

The state-independent reasoning is an important area of information processing by humans, processing independent of human believes.

State dependent quantum logic Now, let couple "quantum Platonic calculus" to the states of mind - mental states.

Then, for some states, the distributivity law holds true

$$P\psi = [(P \wedge Q) \vee (P \wedge \overline{Q})]\psi \tag{13}$$

or

$$\psi = \mathrm{com}(\mathrm{P}, \mathrm{Q})\psi \text{ or } \mathrm{d}(\mathrm{P}, \mathrm{Q})\psi = 0, \tag{14}$$

even if "Platonic equalities" of section are violated.

For three statements,

$$\psi = \mathbf{com}(\mathrm{P}, \mathrm{Q}, \mathrm{R})\psi \text{ or } \mathbf{d}(\mathrm{P}, \mathrm{Q}, \mathrm{R})\psi = 0, \tag{15}$$

Consider the lattice $\mathcal{P}(P, Q, R)$. If condition (15) holds, then lattice $\mathcal{P}(P, Q, R)$ is distributive for the state ψ :

$$X \wedge (Y \vee Z)\psi = [(X \wedge Y) \vee (X \wedge Z)]\psi, \tag{16}$$

where $X, Y, Z = P, Q, R, \overline{P}, \overline{Q}, \overline{R}$.

We set

$$L_{Q,P,R} = \{\psi \in H : d(P, Q, R)\psi = 0\},$$

the kernel of the operator $d(P, Q, R)$**.**

This is a linear subspace of H**. We call it** *the distributivity subspace* **of the lattice** $\mathcal{P}(P, Q, R)$**.**

For states from this subspace, logic of reasoning is classical. We remark that such classicality is the delicate issue.

Logic of propositions $\mathcal{P}(P, Q, R)$ **can be nonclassical, i.e.,** $d(P, Q, R) = I - \mathrm{com}(\mathrm{P}, \mathrm{Q}, \mathrm{R})$ **can be nonzero.**

But, for states from $L_{Q,P,R}$**, reasoning is classical - the distributivity law holds true.**

Commutativity We define state dependent commutativity as

$$[P, Q]\psi = 0; \tag{17}$$

for a triple of propositions, it is defined as pairwise state-commutativity. Condition $\psi \in L(P, Q, R)$ is equivalent to the ψ-commutativity (17).

Quantum instruments The space of linear Hermitian operators in H is linear space over real numbers.

We consider linear operators acting in it, *superoperators*. A superoperator is called positive if it maps the set of positive semi-definite operators into itself.

Any map $x \to \mathcal{I}_A(x)$, where for each x, the map $\mathcal{I}_A(x)$ is a positive superoperator is called *quantum instrument.* It represents one of measurement procedures of an observable A.

The probability for the output $A = x$ is given by the Born rule in the form

$$\Pr\{A = x \| \rho\} = \mathrm{Tr}\,[\mathcal{I}_A(x)\rho]. \tag{18}$$

Measurement with the output $A = x$ generates the state-update by transformation

$$\rho \to \rho_x = \frac{\mathcal{I}_A(x)\rho}{Tr\mathcal{I}_A(x)\rho}. \tag{19}$$

An observable A can be measured by a variety of instruments generating the same probability distribution, but different state updates.

Quantum instruments of the projection-type Let

$$\mathcal{I}_A(x)\rho = P\rho P$$

where P is a projection. Such instrument is called projection instrument-

RRE as experimental test of distributivity of human logic We consider two projections P, Q, and their projective instruments $\mathcal{I}_P$ and $\mathcal{I}_Q$, and a state vector ψ.

Denote their output probability by

$$p(Xx, Yy, Zz, ..) = \mathrm{Tr}(\cdots \mathcal{I}_Z(z)\mathcal{I}_Y(y)\mathcal{I}_X(x)|\psi\rangle\langle\psi|)$$

for $X, Y, Z \in \{P, Q\}$ and $x, y, z \in \{0, 1\}$. Then we have

$$p(Xx, Yy, Zz, ..) = \|\cdots Z^{(z)}Y^{(y)}X^{(x)}\psi\|^2,$$

where $X^{(0)} = X^{\perp}$ and $X^{(1)} = X$ for all $X \in \{P, Q\}$.

The instruments $\mathcal{I}_P$ and $\mathcal{I}_Q$ show the repeatability, i.e.,

$$p(Px, Px) = p(Px)$$

and

$$p(Qx, Qx) = p(Q(x))$$

for any P, Q and ψ. This is the basic property of projective state update.

We say that $\mathcal{I}_P$ and $\mathcal{I}_Q$ show *RRE (the response replicability effect) in* ψ iff

$$p(Px, Qy, Px) = p(Px, Qy)$$

and

$$p(Qx, Py, Qx) = p(Qx, Py)$$

.

RRE concerns correlations for the answers to sequential questions.

Suppose that after answering the A-question with the "yes", Alice is asked another question B, and gives an answer to it.

And then she is asked A again. In the social opinion pools and other natural decision making experiments, Alice would definitely repeat her original answer to A, "yes".

This is $A-B-A$ response replicability. (In the absence of B-question, we get $A - A$ replicability).

The combination of $A-B-A$ and $B-A-B$ replicability forms RRE.

Equivalence of distributivity and RRE The following theorem holds:

Theorem 0.1. *The projective instruments of* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *show RRE in a state* $\boldsymbol{\psi}$ *if and only if* $\mathbf{com}(\boldsymbol{P}, \boldsymbol{Q})\boldsymbol{\psi} = \boldsymbol{\psi}$.

From the above theorem, we can test the distributivity of human logic (or by Theorem A, commutativity of projectors P and Q) in a given state ψ by two projective instruments $\mathcal{I}_P$ and $\mathcal{I}_Q$ namely, P and Q commute in ψ if and only if

$$p(P1, Q1, P1) = p(P1, Q1), \tag{20}$$

$$p(P1, Q0, P1) = p(P1, Q0), \tag{21}$$

$$p(P0, Q1, P0) = p(P0, Q1), \tag{22}$$

$$p(P0, Q0, P0) = p(P0, Q0), \tag{23}$$

$$p(Q1, P1, Q1) = p(Q1, P1), \tag{24}$$

$$p(Q1, P0, Q1) = p(Q1, P0), \tag{25}$$

$$p(Q0, P1, Q0) = p(Q0, P1), \tag{26}$$

$$p(Q0, P0, Q0) = p(Q0, P0). \tag{27}$$

Towards testing distributivity of human logic In this talk, we highlight the role of the distributivity law in human reasoning.

We couple violations of classical logic with violation of distributivity.

Our thoery provides (really unexpected) possibility to dive into the deepest level of human information processing.

RRE can be checked experimentally, see Eqs. (20)-(27).

We hope that coupling of RRE with logic of human reasoning will stimulate psychologists to perform new experiments.

In the light of paper

Busemeyer, J. and Wang, (2017). Is there a problem with quantum models of psychological measurements? *PLOS ONE*, *12*(11), e0187733.

analysis of the methodology and design should precede experiment.

By finding experimental violation of one of Eqs. (20)–(23), experimenters can conclude that

- **either the distributivity law is violated (for the state ψ prepared for the experiment),**
- **or the state update generated by observations cannot be described straightforwardly as orthogonal projection.**

We stress that *the projective-state update implies the classical Bayesian update of probability* [?, ?] **and the use of Bayesian inference in reasoning. Non-projective instruments generate the non-Bayesian state updates and new inference procedures.**

Heuristically it is clear that RRE is very common in human decision making as well as the use of Bayesian update.

Regarding the latter, there is not so much experimental evidence of violation of the classical Bayes rule in human decision making.

Thus, we can conclude that generally humans proceed with classical logic and do not violate distributivity. Of course, in the absence of the real experimental results this is still just speculation.

What is about "fast thinking?" We can guess (but only guess!) that here humans use the projective state update, because it is the simplest for realization.

We still do not know how the brain realizes the mental state update at the neural level. We only know that brain's functioning is based on processing of electric signals generated in neural networks. In engineering of signal processing, the projection operation is widely used and its realization is simple, both algorithmically and technically.

This can serve as an argument in favor of the conjecture that quantum reasoning is based on the projective state update, i.e., that mathematically it is described by projectors.

Fast thinking is processing of in incompatible variables (without respecting the distributivity law). RRE has to be violated. Thus, we suggest to search violations of RRE in experiments exploring fast thinking.

Towards testing incompatibility of mental observables It is difficult if possible at all to prove compatibility (incompatibility) of mental observables in the theoretical framewor.

It seems that it can be determined only experimentally.

Since commutativity is equivalent to distributivity, the RRE-test can be used as well for checking compatibility (incompatibility) of projective type observables.

QOE: question order effect

QOE is an effect of the dependence of the sequential joint probability distribution of answers on the questions' order:

$$p_{AB} \neq p_{BA}.$$

Moore, D. W. (2002). Measuring new types of question-order effects. *Public Opin. Q.*, *60*, **80-91.**

Its quantum-like modeling with projective instruments:

Wang, Z. and Busemeyer, J. R.: A quantum question order model supported by empirical tests of an a priori and precise prediction. Top. Cogn. Sc. 5, 689–710 (2013)

Impossibility of description by projective instruments of combination QOE+RRE for one concrete state The previous considerations were done in the state-dependent framework.

This gives the possibility to improve essentially the basic result on the impossibility to describe combination of QOE and RRE by projective type instruments.

Khrennikov, A., Basieva, I., Dzhafarov, E. N. and Busemeyer, J. R. (2014). Quantum models for psychological measurements: An unsolved problem. *PLOS ONE*, *9*, Art. e110909.

This no-go theorem was formulated under the following *stability assumption*:

"If ψ is a possible initial state vector for a given measurement sequence in an n-dimensional Hilbert space, then there is an open ball $B_r(\psi)$ centered at ψ with a sufficiently small radius $r > 0$, such that any vector $\psi + \delta$ in this ball, normalized by its length $\|\psi+\delta\|$, is also a possible initial state vector for this measurement sequence."

Now, we can omit this stability condition and consider just one fixed state ψ.

If QOE+RRE holds for this state, then measurements cannot be described by projective instruments.

Combination of QOE+RRE+QQ-equality with quantum instruments of non-projective type was modeled in articles:

Ozawa, M. and Khrennikov, A.: Application of theory of quantum instruments to psychology: Combination of question order effect with response replicability effect. Entropy, 22(1), 37 (2020) 1-9436.

Ozawa, M. and Khrennikov, A.: Modeling combination of question order effect, response replicability effect, and QQ-equality with quantum instruments. J. Math. Psych. 100, 102491 (2021)

Concluding remarks We discuss the conjecture that quantum logic is a tool of human reasoning; the brain functioning includes the special system for information processing based on quantum logic, the QL-system.

The role of violation of distributivity is highlighted.

As is shown (Theorem A), distributivity is equivalent to compatibility (and in the quantum formalism, to commutativity of operators).

We point out that reasoning with compatible propositions implies joint assignment of values to these propositions. Such reasoning is consumes a lot of computational resources.

The state-dependent character of quantum reasoning is emphasized. We think that state-dependent modeling of quantum reasoning is especially important for applications to cognition and psychology.

Our present study is closely coupled to the previous research on quantum-like modeling of QOE and RRE. The role of RRE was highlighted through coupling to quantum reasoning, its (non-)distributivity and using (in)compatible logical variables.

We proposed the experimental test for RRE and it can be considered as a test of distributivity-compatibility under the assumption of projective type representation of mental observables.

Finally, we improved the no-go theorem on the impossibility of combination of QOE and RRE.

We hope that this paper would attract attention of psychologists and experts in brain studies to quantum logic conjecture for human reasoning.

In experimental research, coupling of RRE with the basics of quantum reasoning would stimulate its further testing.