Quantum Observables, Contextual Boolean Algebras, and Bayesian Rationality in Decision Trees

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Outline

Preliminaries

Normative Theory of Behaviour in Decision Trees

Research Question

Review: Kolmogorov's Definition of Probability

Two Examples

A Contextual Multi-Measurable Space

A Contextual Multi-Probability Space

Conclusion: Probabilistic Foundations of Quantum Mechanics?

Patrick Suppes, 1922–2014, “Scientific Philosopher”

U.S. National Medal of Science, 1990

Selected Works on Quantum Logic by Patrick Suppes

- (1961) “Probability Concepts in Quantum Mechanics”
  Philosophy of Science 28 (4): 378–389.
- (1966) “The Probabilistic Argument
  for a Non-classical Logic of Quantum Mechanics”
  Philosophy of Science 33 (1/2): 14–21.
- (ed.) (1976) *Logic and Probability in Quantum Mechanics*
  (Dordrecht: Reidel)

Understanding My Title

- In an n-dimensional system, a quantum observable is a self-adjoint operator (or Hermitian matrix) defined on the n-dimensional Hilbert space $\mathbb{C}^n$ of wave vectors.
- A contextual Boolean algebra on $\mathbb{C}^n$ is based on the family of component linear subspaces specified in an orthogonal decomposition of $\mathbb{C}^n$.
- Given the space $\mathbb{C}^n$, there is a bijection between:
  (i) quantum observables;
  (ii) contextually measurable functions.
- A consequentialist decision tree has payoffs at terminal nodes replaced by consequences in a specified domain.
- Bayesian rationality in any finite consequentialist decision tree is identified with choosing actions at each decision node which together maximize the subjective expected utility of the risky and uncertain consequences which arise in each continuation subtree that starts at a decision node.

Subjective Expected Utility with Subjective Probability

Leonard J. Savage *The Foundations of Statistics*

Key ideas of (uncertain) states of the world, consequences, and acts as mappings from states to consequences

Axioms which imply Bayesian rationality — meaning that acts should be chosen in order to maximize subjectively expected utility — so called because it is calculated using a subjective probability attached to each state of the world.

Frank J. Anscombe and Robert J. Aumann (1963) "A Definition of Subjective Probability" *Annals of Mathematical Statistics* 34: 199–205.

To supplement the "horse lotteries" considered by Savage, Anscombe and Aumann introduce "roulette lotteries" with specified "objective" (or hypothetical) probabilities.

Consequentialist Decision Trees: Definition

A consequentialist decision tree
is a (finite) "rooted" acyclic directed graph
with four kinds of node:

1. decision nodes, where the decision maker
   must choose one of a finite set of decisions,
   each of which leads to a unique node that comes next;
2. chance nodes, where a roulette lottery with specified
   hypothetical (or "objective") strictly positive probabilities
   determines randomly which node comes next;
3. event nodes, where a horse lottery determines:
   - which node comes next;
   - how the event consisting of uncertain states of the world
     that remain possible in the relevant continuation subtree
     gets partitioned into pairwise disjoint events;
4. terminal nodes, which get mapped to elements
   of the fixed consequence domain.

Consequentialist Decision Trees in Perspective

In effect, the work by Savage and by Anscombe and Aumann (like many others) considers only “trivial” decision trees in which the unique decision node is at the ”root”.

So really they have reduced:

- a consequentialist decision tree, which is a one-person game in extensive form with payoffs replaced by consequences in the specified consequence domain;
- to its strategic or normal form, which is a tree with only one decision node that offers just one opportunity to choose a decision strategy specifying what to do at every decision node of the original decision tree.

Consequentialist Decision Trees as One-Person Games

John von Neumann (1928) "Zur Theorie der Gesellschaftsspiele"
Mathematische Annalen 100: 295–320.

Von Neumann claims that this reduction to strategic form
loses no generality.

That is, the set of consequences
that can result from different players' possible strategy profiles
should be identical in games whose strategic forms are identical.

Even when prescribing what players should do
rather than describing what they actually should do,
modern game theory disagrees, except for:

1. single-person decision trees,
   as in the theory of Bayesian rationality considered here;
2. two-person zero sum games — the only ones
   with more than one player for which modern game theorists
   still regard von Neumann's solution as satisfactory.

Subjective Expected Utility Maximization in Trees

My previous work on prescriptive decision theory in trees:

"Dynamic Restrictions on Metastatic Choice"
Economica 44 (1977), 337–350.

"Consequentialist Foundations for Expected Utility"
Theory and Decision 25 (1988), 25–78.

Parts of two chapters
for the *Handbook of Utility Theory, Vol. 1: Principles*
(co-edited with Salvador Barberà and Christian Seidl;
Kluwer Academic Publishers, 1998) on:

- "Objective Expected Utility: A Consequentialist Perspective" ch. 5, pp. 145–211;
- "Subjective Expected Utility" ch. 6, pp. 213–271.

"Prerationality as Avoiding Predictably Regrettable Consequences"
Revue économique 73 (2022), 943–976.

Prerational Preferences

Consider a complete family, one for each possible event, of conditional base binary preference relations over the domain of Anscombe–Aumann (or A–A) consequence lotteries — which, by definition, are horse lotteries whose "prize" in each uncertain state of the world is a roulette lottery.

This complete family is said to be prerational just in case there exists a behaviour rule that:

- is defined at every decision node of any finite decision tree, including any continuation decision "subtree", with consequences in the specified domain;
- always leads to a choice set of A–A consequence lotteries which is explicable as avoiding, in all predictable circumstances, any "regrettable" A–A lottery that should not be chosen from the feasible set of consequence lotteries which the decision tree makes possible.

Implications of Prerationality

Theorem

Suppose the conditional base relation in each state of the world is non-trivial, continuous in probabilities over Marschak triangles, and satisfies generalized state independence.

Then the complete family of conditional base relations for all possible different events is prerational if and only if it is refined Bayesian rational — i.e., all subjective probabiliities are strictly positive.

Main idea of proof: consider continuation decision subtrees $T_{\geq n}$ whose set of nodes consists of all nodes of the original tree T that follow n, a specified node of T.

Moreover:

1. node n is the only decision node of $T_{\geq n}$;
2. the decision at node n is between two possible actions, precisely, resulting in two different ensuing Anscombe–Aumann consequence lotteries.

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Quantum Decision Trees with Quantum Nodes

Recall that a (finite) decision tree has four kinds of node:

1. decision nodes;
2. chance nodes (with roulette lotteries);
3. event nodes (with horse lotteries).
4. terminal nodes, where consequences are determined;

To allow for uncertain outcomes of quantum phenomena,
perhaps we need quantum decision trees
with a fifth kind of node
— namely quantum nodes where some quantum experiment determines the outcome of a "quantum lottery" in the form of a measured random value of some "quantum observable".

Describing Von Neumann Quantum Experiments

In an n-dimensional space determined at the initial node, a simple von Neumann quantum experiment has ingredients:

1. the projective Hilbert space $\mathcal{P} = (\mathbb{C}^n \setminus \{\mathbf{0}\})/\sim$ of equivalence classes of wave vectors $\boldsymbol{\psi} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where

$$\boldsymbol{\psi} \sim \tilde{\boldsymbol{\psi}} \iff \exists c \in \mathbb{C} \setminus \{0\} : \tilde{\boldsymbol{\psi}} = c\boldsymbol{\psi}$$

2. an initial time t_0 and an initial wave vector $\boldsymbol{\psi}_0 \in \mathcal{P}$;
3. a constant self-adjoint Hamiltonian energy operator H defined on $\mathbb{C}^n$;
4. an observation time $t_1 \geq t_0$ which, together with the Schrödinger wave equation $\dot{\boldsymbol{\psi}} = -\mathrm{i}\, H\boldsymbol{\psi}$ (in normalized units that remove Planck's constant) determine the wave vector $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 \exp[-\mathrm{i}\, H(t_1 - t_0)]$;
5. a quantum observable in the form of a self-adjoint operator $\mathbf{A}$ defined on $\mathbb{C}^n$.

Research Question: Can Quantum Nodes Be Reduced?

Claim

Under precise conditions still being investigated,
any quantum node in a quantum decision tree
can be reduced to a three-stage process
that compounds three different kinds of node:

1. *a decision node (possibly trivial) that determines a quantum experiment with a specified Boolean algebra based on an orthogonal decomposition of* $\mathbb{C}^n$*;*
2. *a "quantum event" node, where a horse lottery with subjective probabilities specified by a density operator — or equivalently, by a Bayesian prior probability distribution — determines a wave vector for the quantum experiment;*
3. *a "quantum chance" node where, given the wave vector, a "quantum roulette lottery" determines the magnitude of the relevant random "quantum observable", with "objective" probabilities specified by Born's rule.*

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Boolean Algebras, σ-Algebras, and Measurable Spaces

Definition

1. Given any set S, the power set $\mathcal{P}(S) = 2^S$ of S consists of all its subsets.
2. The family $\mathcal{A} \subseteq \mathcal{P}(S)$ is a Boolean algebra on S just in case
   - $\emptyset \in \mathcal{A}$;
   - $A \in \mathcal{A}$ implies that the complement $S \setminus A \in \mathcal{A}$;
   - if A, B lie in $\mathcal{A}$, then the union $A \cup B \in \mathcal{A}$.
3. The family $\Sigma \subseteq \mathcal{P}(S)$ is a σ-algebra just in case it is a Boolean algebra with the following stronger property: whenever $\{A_n\}_{n=1}^{\infty}$ is a countably infinite family of sets in Σ, then their union $\cup_{n=1}^{\infty} A_n \in \Sigma$.
4. The pair (S, Σ) is a measurable space just in case Σ is a σ-algebra on the sample space S.

Remark

If $\mathcal{A} \subseteq \mathcal{P}(S)$ is a Boolean algebra on S, then $S \in \mathcal{A}$.

The Sigma-Algebra Generated by a Partition

Proposition

If $\{\Sigma_f\}_{f \in F}$ is any indexed family of σ-algebras on S, then their intersection $\cap_{f \in F} \Sigma_f$ is also a σ-algebra on S.

Definition

1. Given any family $\mathcal{F} \subset \mathcal{P}(S)$ of subsets of S, the σ-algebra $\sigma(\mathcal{F})$ on S generated by $\mathcal{F}$ is the intersection of all the σ-algebras $\Sigma \supseteq \mathcal{F}$ — i.e., the smallest σ-algebra that includes all the sets in $\mathcal{F}$.
2. The family $\mathfrak{P} \subset \mathcal{P}(S)$ of non-empty subsets of S is a partition of S just in case its different members are pairwise disjoint cells, with $\cup_{E \in \mathfrak{P}} E = S$.

Lemma

The σ-algebra generated by any partition $\mathfrak{P}$ of S satisfies $\sigma(\mathfrak{P}) = \mathcal{P}(\mathfrak{P}) = 2^{\mathfrak{P}}$, which is the power set consisting of all subsets of the set $\mathfrak{P}$ of cells in the partition.

Probability Measures and Probability Spaces

Fix a measurable space (S, Σ),
where S is a set of unknown states of the world.

Then Σ is a σ-algebra of unknown events.

Definition

- A probability measure on the measurable space (S, Σ) is a function $\Sigma \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ satisfying: (i) $\mathbb{P}(S) = 1$; (ii) for any countable collection $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint events in Σ, one has countable additivity — i.e. $\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$.
- A probability space is a triple $(S, \Sigma, \mathbb{P})$ where:
  1. S is the state space;
  2. Σ is a σ-algebra on S, making (S, Σ) a measurable space;
  3. $\Sigma \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ is a probability measure on (S, Σ).

Measurable Functions and Random Variables

Definition

- The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is the smallest σ-algebra that includes every open interval of $\mathbb{R}$.
- Given any measurable space (S, Σ), the mapping $S \ni s \mapsto f(s) \in \mathbb{R}$ is measurable just in case, for each Borel set $B \in \mathcal{B}$, its inverse image $f^{-1}(B) := \{s \in S \mid f(s) \in B\}$ satisfies $f^{-1}(B) \in \Sigma$.
- Given any probability space $(S, \Sigma, \mathbb{P})$, the mapping $S \ni s \mapsto f(s) \in \mathbb{R}$ is a random variable just in case it is measurable as a mapping defined on (S, Σ).
- Any random variable $S \ni s \mapsto f(s) \in \mathbb{R}$ on $(S, \Sigma, \mathbb{P})$ has a cumulative distribution function $\mathbb{R} \ni x \mapsto F(x) \in [0, 1]$ defined for all $x \in \mathbb{R}$ by $F(x) := \mathbb{P}\left(f^{-1}(-\infty, x]\right)$.

The Joint Distribution of Two Random Variables

Remark

Two random variables $X, Y : S \to \mathbb{R}$ have a joint distribution specified by the cumulative joint distribution function

$$\mathbb{R}^2 \ni (x, y) \mapsto F_{XY}(x, y) \in \mathbb{R}$$

if and only if there exist a common σ-algebra Σ on S,
and a common probability measure $\mathbb{P}$ on (S, Σ),
such that both X and Y are measurable
on the measure space (S, Σ).

In this case $F_{XY}(x, y) = \mathbb{P}\left(X^{-1}(-\infty, x] \cap Y^{-1}(-\infty, y]\right)$.

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The Three Sibling Example

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Related Papers by Khrennikov and Hess

Andrei Y. Khrennikov (2015)
"Two-Slit Experiment: Quantum and Classical Probabilities"
Physica Scripta 90 (7): 074017

Cites related work by V.I. and Olga Man′ko

Andrei Y. Khrennikov (2015) "CHSH Inequality:
Quantum Probabilities as Classical Conditional Probabilities"
Foundations of Physics 45: 711–725.

Karl Hess (2020) "Kolmogorov's Probability Spaces
for 'Entangled' Data-Subsets of EPRB Experiments:
No Violation of Einstein's Separation Principle"
Journal of Modern Physics, 11: 683–702.

Karl Hess (2021) "What Do Bell-Tests Prove?
A Detailed Critique of Clauser–Horne–Shimony–Holt
Including Counterexamples"
Journal of Modern Physics, 12: 1219–1236.

Professor Khrennikov: Earlier Work

Andrei Y. Khrennikov (2003)
"Contextual Viewpoint to Quantum Stochastics"
Journal of Mathematical Physics 44(6): 2471–2478.

Andrei Y. Khrennikov (2008)
"EPR–Bohm Experiment and Bell's Inequality:
Quantum Physics Meets Probability Theory"
Theoretical and Mathematical Physics 157: 1448–1460.

David Avis, Paul Fischer, Astrid Hilbert,
and Andrei Khrennikov (2009)
"Single, Complete, Probability Spaces
Consistent with EPR–Bohm–Bell Experimental Data"
In: *Foundations of Probability and Physics-5*,
AIP Conference Proceedings, 1101, 294–301.

Feynman's Contention, I

Feynman, Richard P. (1951)
"The Concept of Probability in Quantum Mechanics"
in Jerzy Neyman (ed.) *Second Berkeley Symposium on Mathematical Statistics and Probability*, (University of California Press: Berkeley, CA), pp. 533–541.

Feynman, Richard P., Robert B. Leighton, and Matthew Sands (1964) *The Feynman Lectures on Physics, Volume III: Quantum Mechanics* (Addison Wesley: Reading, MA).

The usual additivity condition for probability requires that, whenever the two events E and E' are disjoint, and the three probabilities $\mathbb{P}(E)$, $\mathbb{P}(E')$, and $\mathbb{P}(E \cup E')$ are all well defined, then $\mathbb{P}(E \cup E') = \mathbb{P}(E) + \mathbb{P}(E')$.

"Quantum probability" may not satisfy this requirement.

Feynman's Contention, II

As Feynman (1951, p. 533) writes:

. . . far more fundamental was the discovery
that in nature the laws of combining probabilities were not
those of the classical probability theory
. . . you may be delighted to learn that Nature
with her infinite imagination has found
another set of principles for determining probabilities;
a set . . . which nevertheless does not lead
to logical inconsistencies.

What is changed, and changed radically,
is the method of calculating probabilities.

The Mystery of the Double-Slit Experiment

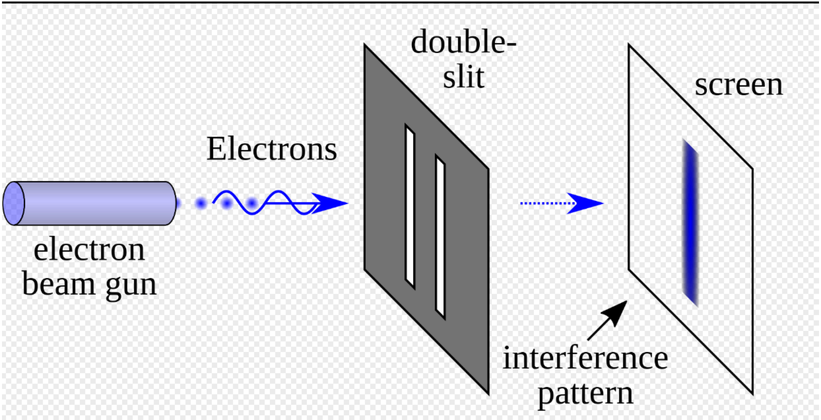
In 1801 Thomas Young had used a celebrated double-slit experiment to demonstrate the wave nature of light.

Feynman (1951) wrote at length
about a quantum version of this experiment.

Later, in their famous series of published lectures,
Feynman *et al.* (1964, pp. 1–2) justified this choice with the claim:

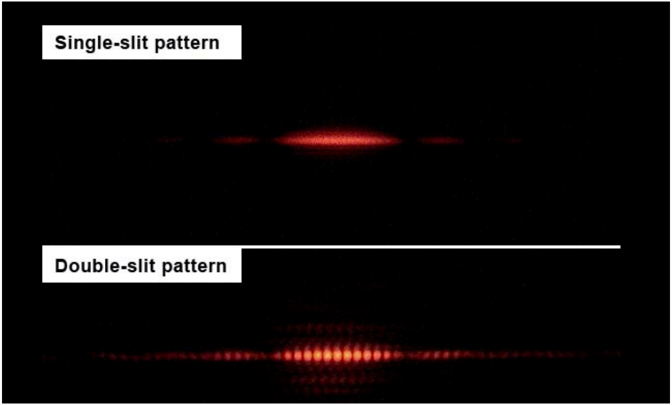
> *We choose to examine a phenomenon which is impossible,*
> *absolutely impossible, to explain in any classical way,*
> *and which has in it the heart of quantum mechanics.*
> *In reality, it contains the only mystery.*

Double-Slit Experiment Illustrated, I



Source:
https://en.wikipedia.org/wiki/Double-slit_experiment

Double-Slit Experiment Illustrated, II



Source:
https://en.wikipedia.org/wiki/Double-slit_experiment

Description of the Double-Slit Experiment

Young's 1801 experiment involved
what appeared to be constant continuous light beams.

Feynman's quantum version involves a beam of discrete electrons, though it could also be other subatomic particles.

- When only one slit is open, which is known to be either L or R, let the two functions $f_L(x, y)$ and $f_R(x, y)$ denote both:
  - **Young** the intensities of the observed light beams;
  - **Feynman** the probability density functions that describe the random impacts of particles on the rear screen.
- When both slits are open, let $f_{LR}(x, y)$ be the corresponding intensity/probability density functions in both versions of the experiment.

Which Slit Has a Particle Passed Through?

When both slits are open, suppose the conditional probabilities that a particle will be detected at the back screen given that it has passed through slits L and R are π_L and π_R respectively, with $\pi_L, \pi_R \geq 0$ and $\pi_L + \pi_R = 1$.

Then the expected density for particles reaching the back screen would be $f_{LR}(x, y) = \pi_L f_L(x, y) + \pi_R f_R(x, y)$, a convex combination of the two functions $f_L(x, y)$ and $f_R(x, y)$.

If the equation $\pi_L f_L(x, y) + \pi_R f_R(x, y) = f_{LR}(x, y)$ were observed to hold, then the probabilities π_L and π_R could be inferred.

Wave Interference

But the equation $f_{LR}(x, y) = \pi_L f_L(x, y) + \pi_R f_R(x, y)$
is manifestly contradicted by,
amongst other things, the empirical observation
that there exist positions (x, y) on the back screen
where $f_{LR}(x, y) > \max\{f_L(x, y), f_R(x, y)\}$.

The inference generally drawn
is that this observed interference effect
contradicts the laws of probability.

Indeed, the conditional probabilities π_L and π_R
seem to be not even defined.

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Vorob′ev's Example

Nikolai N. Vorob′ev (1962)
"Consistent Families of Measures and Their Extensions"
Theory of Probability and its Applications 7: 147–163.

Antecedent:
George Boole (1862) "On the Theory of Probabilities"
Philosophical Transactions of the Royal Society of London 152: 225–252.

A Tale of Three Sibling Sports Enthusiasts

The three siblings Xavier (X), Yvonne (Y) and Zoë (Z)
have only two season tickets for their local sports team.

They take it in turns for one of them to miss any home game.

Whichever pair attends chooses to wear
either Red (R) or Green (G) coloured clothing.

Yvonne and Zoë are identical twins.

If either accompanies Xavier, they will wear his chosen colour.

But if the twins go together,
they wear different colours so they can be told apart.

Three Pairs of Random Variables

Given the triple $(x, y, z) \in \{R, G\}^3$
of the three siblings' clothing choices,
the colours that we actually see, depending on the context,
are different pairs $\{R, G\}^2$ satisfying:

- both pairs (X, Y) and (X, Z) are perfectly correlated;
- but the pair (Y, Z) is perfectly anti-correlated.

These correlations are obviously inconsistent
with any one probability distribution on $\{R, G\}^3$.

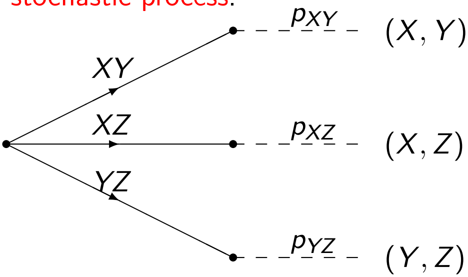
Are Such Correlations Possible?

Are such "inconsistent" pairwise correlations logically possible?

- No if we limit ourselves to one joint probability distribution on $\Omega_{XYZ} := \{R, G\}^3 = \{R, G\}_X \times \{R, G\}_Y \times \{R, G\}_Z$.
- Yes if we introduce a fourth variable $c \in C = \{XY, XZ, YZ\}$ indicating the context in which observations are made.

  Its value determines which two of the first 3 variables we see — in our example, which two siblings appear at a game.

Then our observations can emerge from a single stochastic process.



A Multi-Measurable Space

Depending on the context, the sample space

$$\Omega_{XYZ} := \{R, G\}^3 = \{R, G\}_X \times \{R, G\}_Y \times \{R, G\}_Z$$

can be made into a measurable space by appending any one of the three different contextual products of two Boolean algebras:

$$\mathcal{B}_{XY} := \mathcal{P}(\{R_X, G_X\} \times \{R_Y, G_Y\}) \otimes \{\emptyset, \{R_Z, G_Z\}\}$$
$$\mathcal{B}_{XZ} := \mathcal{P}(\{R_X, G_X\} \times \{R_Z, G_Z\}) \otimes \{\emptyset, \{R_Y, G_Y\}\}$$
$$\mathcal{B}_{YZ} := \mathcal{P}(\{R_Y, G_Y\} \times \{R_Z, G_Z\}) \otimes \{\emptyset, \{R_X, G_X\}\}$$

In each case, R and G are indistinguishable for one missing sibling.

Following Vorob′ev (1962),
we can also equip Ω_{XYZ} with all three Boolean algebras
so it becomes the multi-measurable space $(\Omega_{XYZ}, \mathcal{B}_{XY}, \mathcal{B}_{XZ}, \mathcal{B}_{YZ})$.

Weird Correlations Explained

Suppose that for each context $c \in C$,
the observed pair of colours in Ω_c is determined
by a contextual probability density function π_c.

Then the weirdly correlated observations occur
if and only if there exist three constants $\alpha, \beta, \gamma \in [0, 1]$
such that the three density functions π_c ($c \in C$) are as below.

π_{XY}	R_Y	B_Y
R_X	α	0
B_X	0	$1 - \alpha$

π_{XZ}	R_Z	B_Z
R_X	β	0
B_X	0	$1 - \beta$

π_{YZ}	R_Z	B_Z
R_Y	0	γ
B_Y	$1 - \gamma$	0

As promised, both pairs (x, y) and (x, z) of random variables
are perfectly correlated,
yet y and z are perfectly anti-correlated.

Constructing a Multi-Probability Space

Starting with the multi-measurable space $(\Omega_{XYZ}, \mathcal{B}_{XY}, \mathcal{B}_{XZ}, \mathcal{B}_{YZ})$,
for each context $c \in C = \{XY, XZ, YZ\}$,
the contextual measurable space $(\Omega_{XYZ}, \mathcal{B}_c)$
has become the contextual probability space $(\Omega_{XYZ}, \mathcal{B}_c, \pi_c)$.

Thus, we have three contextual probability spaces.

All are based on the same sample space Ω_{XYZ}.

Putting all three contextual probability spaces together
results in the single multi-probability space $(\Omega_{XYZ}, (\mathcal{B}_c, \pi_c)_{c \in C})$.

This is an example of what Vorob′ev (1962) described
as a "family of measures".

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Can There Be a Classical Probability Space?

The two-slit experiment has the sample space $\Omega = S \times D$ where:

1. $S = \{L, R\}$ is the set of two slits in the front screen, of which any subset could be open;
2. a bounded subset $D \subset \mathbb{R}^2$ is the domain of possible points of observed impact on the back screen.

To make this a classical probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we might try defining $\mathcal{A}$ as the family of subsets of $2^S \times D$ whose members take the form

$$(\{L\} \times D_L) \cup (\{R\} \times D_R) \cup (\{L, R\} \times D_{L,R})$$

where D_L, D_R, and $D_{L,R}$ are three Borel subsets of D.

But we already saw that no single probability measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ can account for all the observations in the different contexts where either or both slits are open.

We try again with a multi-probability space.

Three Different Contexts

We recognize that:

1. we can ignore the trivial case when neither slit is open, so no impact on the back screen can be observed;
2. the non-empty set of open slits determines three different contexts $k \in K := \{L, R, LR\}$;
3. we should consider a multi-probability space of the form $(\Omega, (\mathcal{A}_k, \mathbb{P}_k)_{k \in K})$, with a separate contextual probability space $(\Omega, \mathcal{A}_k, \mathbb{P}_k)$ for each possible context $k \in K$.

Recall the notation $D \ni (x, y) \mapsto f_k(x, y) \in \mathbb{R}_+$ for the continuous probability density function on D that is relevant for each context $k \in K$.

The Resulting Multi-Probability Space: Two One-Slit Parts

In either of the two contexts where $k = L$ or $k = R$,
it is known which one of the two slits is open.

The associated contextual probability space $(S \times D, \mathcal{A}_k, \mathbb{P}_k)$ has:

1. the σ-algebra $\mathcal{A}_k$ on $S \times D$ whose only non-empty sets take the form $D_k \times \{k\}$ for some Borel set $D_k \subseteq D$;
2. the probability measure $\mathbb{P}_k$ that, for each $D_k \subseteq D$, and so for each $D_k \times \{k\} \in \mathcal{A}_k$, satisfies

$$\mathbb{P}_k(D_k \times \{k\}) = \int_{D_k} f_k(x, y)(\mathrm{d}\, x \times \mathrm{d}\, y)$$

The Resulting Multi-Probability Space: One Two-Slit Part

In the context where $k = LR$, so both slits are open,
the associated contextual probability space $(S \times D, \mathcal{A}_{LR}, \mathbb{P}_{LR})$ has:

1. the σ-algebra $\mathcal{A}_{LR}$ on $S \times D$ whose only non-empty sets take the form $D_{LR} \times \{L, R\}$ for some Borel set $D_{LR} \subseteq D$ (representing the fact that L and R are indistinguishable);
2. the probability measure $\mathbb{P}_{LR}$ that, for each Borel set $D_{LR} \subseteq D$ and so for each $D_{LR} \times \{L, R\} \in \mathcal{A}_{LR}$, satisfies

$$\mathbb{P}_{LR}(D_{LR} \times \{L, R\}) = \int_{D_{LR}} f_{LR}(x, y)(\mathrm{d}\, x \times \mathrm{d}\, y)$$

For the set $K = \{L, R, LR\}$ of three possible contexts,
this completes the construction
of the three-part multi-probability space $(S \times D, (\mathcal{A}_k, \mathbb{P}_k)_{k \in K})$,
each with its own contextual probability
density function $D \ni (x, y) \mapsto f_k(x, y) \in \mathbb{R}_+$ on D.

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Hilbert Space and Adjoint Matrices

The n-dimensional linear space $\mathbb{C}^n$ over the algebraic field $\mathbb{C}$ has as its typical member the column n-vector $\mathbf{x} = (x_i)_{i=1}^n$ whose n components are complex numbers.

The space $\mathbb{C}^n$ becomes a Hilbert space when equipped with the complex-valued inner product which is defined for all pairs $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$ of column n-vectors by $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^n \bar{x}_i y_i$.

This Hilbert space has a real-valued norm $\|\mathbf{x}\| \geq 0$ whose square is defined for all n-vectors $\mathbf{x} = (x_i)_{i=1}^n$ by

$$\|\mathbf{x}\|^2 :=< \mathbf{x}, \mathbf{x} >= \sum_{i=1}^n \bar{x}_i x_i = \sum_{i=1}^n |x_i|^2$$

Definition

The adjoint $\mathbf{A}^*$ of any $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$ is defined as the $n \times m$ transposed conjugate $\mathbf{A}^* = (a_{ij}^*)_{n \times m}$ whose elements satisfy $a_{ij}^* = \bar{a}_{ji}$, implying that $\mathbf{A}^* = \bar{\mathbf{A}}^\top$.

Adjoint Matrices and Inner Product Notation

Remark

Our notation allows the inner product $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} \bar{x}_i y_i$ *of any pair of column* n*-vectors* $\mathbf{x} = (x_i)_{i=1}^{n}$ *and* $\mathbf{y} = (y_i)_{i=1}^{n}$ *to be rewritten more concisely as the* 1×1 *"matrix" product* $\mathbf{x}^* \mathbf{y}$ *of the* $1 \times n$ *adjoint row matrix* $\mathbf{x}^* = ((\bar{x}_i)_{i=1}^{n})^\top = \bar{\mathbf{x}}^\top$ *with the original* $n \times 1$ *column matrix* $\mathbf{y} = (y_i)_{i=1}^{n}$.

Self-Adjoint Matrices and Their Eigenpairs

Definition

- The $n \times n$ matrix $\mathbf{A}$ is self-adjoint just in case $\mathbf{A}^* = \mathbf{A}$.
  (This is an extension to $\mathbb{C}^n$ of symmetric matrices on $\mathbb{R}^n$.)
- The $n \times n$ matrix $\mathbf{U}$ is unitary just in case $\mathbf{U}^* = \mathbf{U}^{-1}$.
  (This is an extension to $\mathbb{C}^n$ of orthogonal matrices on $\mathbb{R}^n$.)
- The pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\mathbf{0}\})$ is an eigenpair of $\mathbf{A}$
  just in case $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$,
  so λ is an eigenvalue and $\mathbf{x} \neq \mathbf{0}$ is an eigenvector.
- The spectrum of a matrix $\mathbf{A}$ is the set $s^{\mathbf{A}}$ of its eigenvalues.

Properties of the Eigenpairs of a Self-Adjoint Matrix

Proposition

Suppose that $\mathbf{A}$ *is any self-adjoint matrix on* $\mathbb{C}^n$
and that $(\lambda, \mathbf{x})$ *with* $\mathbf{x} \neq \mathbf{0}$ *is any eigenpair.*

- *Because* $\mathbf{A} = \mathbf{A}^*$ *one has*

  $$(\lambda - \bar{\lambda})\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) - (\bar{\lambda}\mathbf{x}^*)\mathbf{x} = \mathbf{x}^*(\mathbf{A}\mathbf{x}) - (\mathbf{x}^*\mathbf{A}^*)\mathbf{x} = 0$$

  But $\mathbf{x} \neq \mathbf{0}$ *implies that* $\mathbf{x}^*\mathbf{x} > 0$.
  It follows that $\lambda = \bar{\lambda}$, *so any eigenvalue* λ *is real.*

- *If* $(\mu, \mathbf{y})$ *is any other eigenpair with* $\lambda \neq \mu$,
  because $\mathbf{A} = \mathbf{A}^*$, $\lambda = \bar{\lambda}$, *and* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, *one has*

  $$(\lambda - \mu)\mathbf{x}^*\mathbf{y} = (\bar{\lambda}\mathbf{x}^*)\mathbf{y} - \mathbf{x}^*(\mu\mathbf{y}) = (\mathbf{A}\mathbf{x})^*\mathbf{y} - \mathbf{x}^*(\mathbf{A}\mathbf{y}) = 0$$

  But $\lambda - \mu \neq 0$, *so* $\mathbf{x}^*\mathbf{y} = 0$.
  It follows that any two eigenvectors $\mathbf{x}$ *and* $\mathbf{y}$
  corresponding to different eigenvalues must be orthogonal.

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Orthogonal and One-Dimensional Linear Subspaces of $\mathbb{C}^n$

Definition

- A linear subspace $L \subset \mathbb{C}^n$ is a subset
  that is closed under linear combinations
  — i.e., if $\mathbf{x}, \mathbf{y} \in L$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha \mathbf{x} + \beta \mathbf{y} \in L$.
- Two linear subspaces L and $\tilde{L}$ are orthogonal
  just in case $\mathbf{x}^* \mathbf{y} = 0$ for all $\mathbf{x} \in L$ and $\mathbf{y} \in \tilde{L}$.
- Given any $\mathbf{e} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, let

  $$[\mathbf{e}] := \text{span}(\{\mathbf{e}\}) := \{\mathbf{x} \in \mathbb{C}^n \mid \exists c \in \mathbb{C} : \mathbf{x} = c\mathbf{e}\}$$

  denote the one-dimensional linear subspace of $\mathbb{C}^n$
  that is spanned by $\mathbf{e}$.

Orthogonal Decompositions of $\mathbb{C}^n$: Definition

Definition

The indexed finite family $\mathcal{L}^D = \{L_d\}_{d \in D}$
of mutually orthogonal linear subspaces L_d
is an orthogonal decomposition of $\mathbb{C}^n$
just in case the direct or vector sum

$$\bigoplus_{d \in D} L_d := \left\{ \mathbf{x}^+ \in \mathbb{C}^n \;\middle|\; \forall d \in D; \exists \mathbf{x}_d \in L_d : \mathbf{x}^+ = \sum_{d \in D} \mathbf{x}_d \right\}$$

of all the subspaces L_d in $\mathcal{L}^D$ is equal to the whole of $\mathbb{C}^n$.

Orthogonal Decompositions of $\mathbb{C}^n$: Example

Definition

For each natural number $m \in \mathbb{N}$, define $\mathbb{N}_m := \{1, 2, \dots m\} \subset \mathbb{N}$.

Example

Let $\{\mathbf{e}^{(k)}\}_{k \in \mathbb{N}_n}$ be any orthonormal basis of $\mathbb{C}^n$,
and let $\{M_r\}_{r \in \mathbb{N}_m}$ be any partition of $\mathbb{N}_n$
into m pairwise disjoint non-empty sets.

For each $r \in \mathbb{N}_m$, define L_r as the linear space of dimension $\# M_r$
spanned by the set $\{\mathbf{e}^{(k)} \mid k \in M_r\}$ of basis vectors.

Then $\bigoplus_{r \in \mathbb{N}_m} L_r$ is an orthogonal decomposition into m subspaces.

Except in the special case when $\# M_r = 1$ for all $r \in \mathbb{N}_m$,
this is entirely different
from the orthogonal decomposition $\oplus_{k \in \mathbb{N}_n} L_{[\mathbf{e}^{(k)}]}$ of $\mathbb{C}^n$
into the collection of n one-dimensional subspaces
that are each spanned by one of the basis vectors.

Unitary Transforms of Orthogonal Decompositions

Proposition

If $\bigoplus_{d \in D} L_d$ is an orthogonal decomposition of $\mathbb{C}^n$ then,
given any $n \times n$ unitary matrix $\mathbf{U}$,
so is the unitarily transformed family $\bigoplus_{d \in D} \mathbf{U} L_d$
where, for each $d \in D$, one has

$$\mathbf{U} L_d := \{\mathbf{y} \in \mathbb{C}^n \mid \exists \mathbf{x} \in L_d : \mathbf{y} = \mathbf{U}\mathbf{x}\}$$

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Orthogonal Projection Matrices

Definition

The orthogonal projection $\mathbf{x}_L^\perp$ of any $\mathbf{x} \in \mathbb{C}^n$
onto any linear subspace L of $\mathbb{C}^n$
is the unique closest point of L to $\mathbf{x}$
i.e., it satisfies $\{\mathbf{x}_L^\perp\} := \arg\min_{\mathbf{y} \in L} (\mathbf{x} - \mathbf{y})^*(\mathbf{x} - \mathbf{y})$.

Properties of Orthogonal Projections

Proposition

Let L be any linear subspace of $\mathbb{C}^n$.

1. *For any $\mathbf{x} \in \mathbb{C}^n$, its projection $\mathbf{x}_L^\perp$ is the unique point of L that satisfies $(\mathbf{x} - \mathbf{x}_L^\perp)^*(\mathbf{x}_L^\perp - \mathbf{y}) = 0$ for all $\mathbf{y} \in L$;*
2. *The mapping $\mathbb{C}^n \ni \mathbf{x} \mapsto \mathbf{x}_L^\perp \in L$ is linear, so there exists a projection matrix $\mathbf{\Pi}_L$ such that $\mathbf{\Pi}_L \mathbf{x} = \mathbf{x}_L^\perp$ for all $\mathbf{x} \in \mathbb{C}^n$;*
3. *The projection matrix $\mathbf{\Pi}_L$ satisfies $\mathbf{\Pi}_L^2 = \mathbf{\Pi}_L = \mathbf{\Pi}_L^*$.*
4. *If $\tilde{L}$ is any linear subspace of $\mathbb{C}^n$ that is orthogonal to L, then $\mathbf{\Pi}_L + \mathbf{\Pi}_{\tilde{L}} = \mathbf{\Pi}_{L \oplus \tilde{L}}$.*
5. *If $\{\mathbf{e}^{(k)}\}_{k=1}^m$ is any orthonormal basis of L, then $\mathbf{\Pi}_L = \sum_{k=1}^m \mathbf{\Pi}_{[\mathbf{e}^{(k)}]}$ where each $[\mathbf{e}^{(k)}]$ denotes the one-dimensional subspace spanned by the basis vector $\mathbf{e}^{(k)}$.*

One-Dimensional Orthogonal Projections

Proposition

Given any $\mathbf{e}$ *in the unit sphere* $\mathbb{S}$ *of* $\mathbb{C}^n$*,*
the $n \times n$ *matrix* $\mathbf{P} := \mathbf{e}\mathbf{e}^*$ *is self-adjoint*
and represents the orthogonal projection $\Pi_{[\mathbf{e}]}$ *of* $\mathbb{C}^n$
onto the one-dimensional subspace $[\mathbf{e}] = \text{span}(\{\mathbf{e}\})$*.*

Proof.

For each $\mathbf{e} \in \mathbb{S}$, the $n \times n$ matrix $\mathbf{P} := \mathbf{e}\mathbf{e}^*$ satisfies:

1. $\mathbf{P}^* = (\mathbf{e}\mathbf{e}^*)^* = (\mathbf{e}^*)^*\mathbf{e}^* = \mathbf{e}\mathbf{e}^* = \mathbf{P}$, so $\mathbf{P}$ is self-adjoint;
2. $\mathbf{P}^2 = \mathbf{e}(\mathbf{e}^*\mathbf{e})\mathbf{e}^* = \mathbf{P}$ because $\mathbf{e}^*\mathbf{e} = 1$,
   so $\mathbf{P}$ is an orthogonal projection;
3. $\mathbf{P}\mathbf{x} = \mathbf{e}\mathbf{e}^*\mathbf{x} = c\mathbf{e} \in [\mathbf{e}]$ for all $\mathbf{x} \in \mathbb{C}^n$, where $c = \mathbf{e}^*\mathbf{x} \in \mathbb{C}$,
   so $\mathbf{P}$ is the orthogonal projection $\Pi_{[\mathbf{e}]}$ of $\mathbb{C}^n$ onto $[\mathbf{e}]$. □

Ortho-Partitions and Ortho-Measurability in $\mathbb{C}^n$

Proposition

Given any orthogonal decomposition $\{L_d\}_{d \in D}$ of $\mathbb{C}^n$:

1. *any two different subsets $L_d \setminus \{\mathbf{0}\}$ and $L_{d'} \setminus \{\mathbf{0}\}$ in the finite family $\cup_{d \in D} \{L_d \setminus \{\mathbf{0}\}\}$ are pairwise disjoint;*
2. *the associated family $\{R^D\} \cup [\cup_{d \in D}\{L_d \setminus \{\mathbf{0}\}\}]$ forms a partition of $\mathbb{C}^n$, called the associated ortho-partition $\mathfrak{P}^D$, if and only if R^D is the residual set defined by $R^D := \mathbb{C}^n \setminus \cup_{d \in D}(L_d \setminus \{\mathbf{0}\})$.*

Definition

The ortho-algebra Σ^D generated
by the orthogonal decomposition $\{L_d\}_{d \in D}$ of $\mathbb{C}^n$
is defined as $\sigma(\mathfrak{P}^D)$, the σ-algebra
generated by the ortho-partition $\mathfrak{P}^D = \{R^D\} \cup [\cup_{d \in D}\{L_d \setminus \{\mathbf{0}\}\}]$,
which equals the power set $\mathcal{P}(\mathfrak{P}^D) = 2^{\mathfrak{P}^D}$
of all subsets of the set of cells in the ortho-partition $\mathfrak{P}^D$.

The Spectral Decomposition of a Self-Adjoint Matrix

Given any self-adjoint matrix $\mathbf{A}$ and any $\lambda \in s^{\mathbf{A}}$, let:

1. $E_\lambda := \{\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\} \mid \mathbf{A}\mathbf{x} = \lambda \mathbf{x}\}$
   be the corresponding non-empty set of eigenvectors;
2. $L_\lambda := E_\lambda \cup \{\mathbf{0}\}$ be the corresponding (linear) eigenspace.

Proposition

Any self-adjoint matrix $\mathbf{A}$ has a spectral decomposition of the form $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L_\lambda}$ where $\bigoplus_{\lambda \in s^{\mathbf{A}}} L_\lambda$ is the orthogonal decomposition of $\mathbb{C}^n$ into subspaces $L_\lambda = E_\lambda \cup \{\mathbf{0}\}$ that correspond to the eigenspaces E_λ of $\mathbf{A}$, one for each eigenvalue $\lambda \in s^{\mathbf{A}}$.

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From a Self-Adjoint Matrix to Its Eigen-Pairing

Definition

Given any self-adjoint matrix $\mathbf{A}$
with spectral decomposition $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L_\lambda}$, define:

1. for each eigenvalue $\lambda \in s^{\mathbf{A}}$ and associated eigenspace L_λ:
   - the set $E_\lambda = L_\lambda \setminus \{\mathbf{0}\}$ of corresponding eigenvectors;
   - the indicator function $\mathbb{C}^n \ni \mathbf{x} \mapsto 1_{E_\lambda}(\mathbf{x}) \in \{0, 1\}$
     with the property that $1_{E_\lambda}(\mathbf{x}) = 1 \iff \mathbf{x} \in E_\lambda$;
2. the residual set $R^{\mathbf{A}} := \mathbb{C}^n \setminus \cup_{\lambda \in s^{\mathbf{A}}} E_\lambda$ of n-vectors (including $\mathbf{0}$) that are not eigenvectors of $\mathbf{A}$, for any of its eigenvalues;
3. the one-point extension $\mathbb{R} \cup \{*\}$ of the real line, where $* \notin \mathbb{R}$;
4. the eigen-pairing of $\mathbf{A}$ as the map $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$
   given by $f^{\mathbf{A}}(\mathbf{x}) := \begin{cases} \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, 1_{E_\lambda}(\mathbf{x}) & \text{if } \mathbf{x} \notin R^{\mathbf{A}}; \\ * & \text{if } \mathbf{x} \in R^{\mathbf{A}}. \end{cases}$

Note that $f^{\mathbf{A}}(\mathbf{x}) = \lambda \iff \mathbf{x} \in E_\lambda \iff \mathbf{x} \neq \mathbf{0} \;\&\; \mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.

Making the Eigen-Pairing Measurable

We make the eigen-pairing $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ a measurable function.

1. First, we give the one-point extension $\mathbb{R} \cup \{*\}$ of $\mathbb{R}$, which is the co-domain, the smallest σ-algebra $\sigma(\mathcal{B} \cup \{*\})$ that includes: (i) the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$; (ii) the set $\{*\}$.

   Evidently $\sigma(\mathcal{B} \cup \{*\}) = \cup_{B \in \mathcal{B}} \{B, B \cup \{*\}\}$.
2. Second, we give the domain $\mathbb{C}^n$ the σ-algebra $\Sigma^{\mathbf{A}}$, defined as the smallest σ-algebra $\sigma\left((f^{\mathbf{A}})^{-1}(\sigma(\mathcal{B} \cup \{*\})\right)$ that is large enough to include all sets in the family $\{(f^{\mathbf{A}})^{-1}(Y) \mid Y \in \sigma(\mathcal{B} \cup \{*\})\}$ of inverse images of measurable sets Y in the co-domain $\mathbb{R} \cup \{*\}$.

This constructs $\Sigma^{\mathbf{A}}$ as the smallest σ-algebra which makes $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ measurable as a mapping from the measurable space $(\mathbb{C}^n, \Sigma^{\mathbf{A}})$ to the measurable space $(\mathbb{R} \cup \{*\}, \sigma(\mathcal{B} \cup \{*\}))$.

Characterizing the Two σ-algebras

Theorem

1. *The σ-algebra $\sigma(\mathcal{B} \cup \{*\})$ on the co-domain $\mathbb{R} \cup \{*\}$ is $\cup_{B \in \mathcal{B}}\{B, B \cup \{*\}\}$.*
2. *The σ-algebra $\Sigma^{\mathbf{A}}$ on the domain $\mathbb{C}^n$ is the power set $\mathcal{P}(\mathfrak{P}^{\mathbf{A}})$ consisting of all subsets of the finite set of cells in the ortho-partition $\mathfrak{P}^{\mathbf{A}}$ that is induced by the orthogonal decomposition of $\mathbb{C}^n$ into the different eigenspaces of $\mathbf{A}$.*

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Ortho-Measurable Functions as Self-Adjoint Matrices

Let $\oplus_{d \in D} L_d$ be any finite orthogonal decomposition of $\mathbb{C}^n$,
with residual set $R^D = \mathbb{C}^n \setminus \cup_{d \in D}(L_d \setminus \{\mathbf{0}\})$.

The general function $\mathbb{C}^n \ni \mathbf{x} \mapsto f^D(\mathbf{x}) \in \mathbb{R} \cup \{*\}$
is ortho-measurable w.r.t. $\{L_d\}_{d \in D}$ if and only if,
for each r in $\mathbb{R} \cap f^D(\mathbb{C}^n)$, the real part of the range of f^D,
there exists $D_r \subseteq D$ such that $(f^D)^{-1}(\{r\}) = \cup_{d \in D_r}(L_d \setminus \{\mathbf{0}\})$.

It follows that for all $\mathbf{x} \in \mathbb{C}^n \setminus R^D = \cup_{d \in D}(L_d \setminus \{\mathbf{0}\})$
one has $f^D(\mathbf{x}) = r$ if $\mathbf{x} \in \cup_{d \in D_r}(L_d \setminus \{\mathbf{0}\})$,
and so $f^D(\mathbf{x}) = \sum_{r \in \mathbb{R} \cap f^D(\mathbb{C}^n)} \sum_{d \in D_r} r \, 1_{L_d \setminus \{\mathbf{0}\}}(\mathbf{x})$.

This obviously corresponds to the self-adjoint matrix $\mathbf{A}^D$
defined by the weighted sum $\sum_{r \in \mathbb{R} \cap f^D(\mathbb{C}^n)} \sum_{d \in D_r} r \, \mathbf{\Pi}_{L_d}$
of projections onto the respective linear spaces
which appear in the orthogonal decomposition $\oplus_{d \in D} L_d$.

Main Theorem: From Observable to Measurable Function

Theorem

Corresponding to any quantum observable in the form of a self-adjoint matrix $\mathbf{A}$ *with spectral decomposition* $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \Pi_{L_\lambda}$,
there is a unique function $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ *with*

$$f^{\mathbf{A}}(\mathbf{x}) = \begin{cases} \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, 1_{L_\lambda}(\mathbf{x}) & \text{if } \mathbf{x} \in \cup_{\lambda \in s^{\mathbf{A}}} L_\lambda \\ * & \text{otherwise} \end{cases}$$

that is measurable w.r.t. the spectral ortho-algebra $\Sigma^{\mathbf{A}}$.

Main Theorem: From Measurable Function to Observable

Theorem

Conversely, consider any orthogonal decomposition $\bigoplus_{L \in \mathcal{L}^D} L$ of $\mathbb{C}^n$, any mapping $\mathcal{L}^D \ni L \mapsto \gamma_L \in \mathbb{R}$, and any function $\mathbb{C}^n \ni \mathbf{x} \mapsto g^D(\mathbf{x}) \in \mathbb{R} \cup \{*\}$ with

$$g^D(\mathbf{x}) = \begin{cases} \sum_{L \in \mathcal{L}^D} \gamma_L 1_L(\mathbf{x}) & \text{if } \mathbf{x} \in \cup_{L \in \mathcal{L}^D}(L \setminus \{\mathbf{0}\}) \\ * & \text{otherwise} \end{cases}$$

that is measurable w.r.t. the ortho-algebra Σ^D induced by the orthogonal decomposition.

Then there is a unique corresponding quantum observable in the form of a self-adjoint matrix $\mathbf{A}_g = \sum_{L \in \mathcal{L}^D} \gamma_L \Pi_L$.

A Contextual Multi-Measurable Space

Let $\mathcal{D}$ denote the family of all orthogonal decompositions of $\mathbb{C}^n$.

Then, for each orthogonal decomposition $D \in \mathcal{D}$
and associated ortho-algebra Σ^D,
the pair $(\mathbb{C}^n, \Sigma^D)$ is a measurable space
that depends on the orthogonal decomposition D,
regarded as a context.

Also, the pair $(\mathbb{C}^n, (\Sigma^D)_{D \in \mathcal{D}})$
with the complete family of all ortho-algebras
is a multi-measurable space.

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A Contextual Multi-Probability Space

Pre-Probability Wave Vectors as Pure Quantum States

Density Matrices as Mixed Quantum States

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Definitions of Wave Vector and Basic Ortho-Algebra

Definition

A wave vector or pure quantum state $\boldsymbol{\psi} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ is a non-zero element of the Hilbert space $\mathbb{C}^n$.

Definition

Let $\mathcal{B} = (\mathbf{b}^{(j)})_{j=1}^n$ be any orthonormal basis of $\mathbb{C}^n$.

Given any $j \in \mathbb{N}_n$, let:

1. $L_j := [\mathbf{b}^{(j)}]$ denote the one-dimensional linear space spanned by $\mathbf{b}^{(j)}$;
2. $\mathbf{\Pi}_{L_j}$ denote the $n \times n$ self-adjoint matrix $\mathbf{b}^{(j)}(\mathbf{b}^{(j)})^*$, which represents the projection mapping onto L_j.

Then $D^{\mathcal{B}} := \bigoplus_{j=1}^n L_j$ is a basic orthogonal decomposition of $\mathbb{C}^n$.

1. Let $\mathfrak{P}^{\mathcal{B}}$ denote the associated basic ortho-partition.
2. Let $\Sigma^{\mathcal{B}}$ denote the associated basic ortho-algebra generated by the basic ortho-partition.

Wave Vectors as Pre-Probabilities: Special Case

Consider any basic ortho-algebra $\Sigma^{\mathcal{B}}$
associated with an orthonormal basis $\mathcal{B} = (\mathbf{b}^{(j)})_{j=1}^{n}$ of $\mathbb{C}^n$.

We use the wave vector $\boldsymbol{\psi} \in \mathbb{C}^n \setminus \{0\}$ to construct
a probability measure $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}$ on the measurable space $(\mathbb{C}^n, \Sigma^{\mathcal{B}})$,
according to Born's Rule, treating $\boldsymbol{\psi}$ as a parameter.

This requires that, for each $j \in \mathbb{N}_n$ and $L_j = [\mathbf{b}^{(j)}]$,
the probability $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\})$ of the basic set $L_j \setminus \{\mathbf{0}\} \in \Sigma^{\mathcal{B}}$
is given by the Rayleigh quotient $\boldsymbol{\psi}^* \mathbf{\Pi}_{L_j} \boldsymbol{\psi} / \boldsymbol{\psi}^* \boldsymbol{\psi}$.

But $\mathbf{\Pi}_{L_j} = \mathbf{b}^{(j)} (\mathbf{b}^{(j)})^*$ in this special case, so

$$\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\}) = \boldsymbol{\psi}^* \mathbf{b}^{(j)} (\mathbf{b}^{(j)})^* \boldsymbol{\psi} / \boldsymbol{\psi}^* \boldsymbol{\psi} = |\boldsymbol{\psi}^* \mathbf{b}^{(j)}|^2 / |\boldsymbol{\psi}|^2$$

In the very special case of the canonical orthonormal basis
when $\mathcal{B}$ consists of the columns of the identity matrix,
this reduces to $\mathbb{P}_{\boldsymbol{\psi}}^{\mathcal{B}}(L_j \setminus \{\mathbf{0}\}) = |\psi_j|^2 / |\boldsymbol{\psi}|^2$,
which is the squared modulus rule for calculating probabilities.

Wave Vectors as Pre-Probabilities: General Case

Let $\oplus_{d \in D} L_d$ be any orthogonal decomposition of $\mathbb{C}^n$, with associated ortho-partition $\mathfrak{P}^D$ and ortho-algebra Σ^D.

Born's Rule now requires that the probability $\mathbb{P}^D_\psi(L_d \setminus \{\mathbf{0}\})$ of each non-residual cell $L_d \setminus \{\mathbf{0}\}$ in the ortho-partition $\mathfrak{P}^D$ must equal the value $\psi^* \mathbf{\Pi}_{L_d} \psi|/|\psi|^2$ of the Rayleigh quotient, where now the projection represented by the matrix $\mathbf{\Pi}_{L_d}$ may be onto a space L_d whose dimension exceeds one.

Note that $\sum_{d \in D} \mathbf{\Pi}_{L_d} = \mathbf{I}$ for an orthogonal decomposition, and so

$$\begin{aligned}\sum_{d \in D} \mathbb{P}^D_\psi(L_d \setminus \{\mathbf{0}\}) &= \sum_{d \in D} \psi^* \mathbf{\Pi}_{L_d} \psi / |\psi|^2 \\ &= \psi^* \left(\sum_{d \in D} \mathbf{\Pi}_{L_d} \right) \psi / |\psi|^2 = \psi^* \mathbf{I} \psi / |\psi|^2 = 1\end{aligned}$$

Also $\mathbb{P}^D_\psi(R^D) = 0$ for the residual set $R^D = \mathbb{C}^n \setminus \cup_{d \in D}(L_d \setminus \{\mathbf{0}\})$.

The Expectation in a Pure State of a Quantum Observable

Consider any pure quantum state $\boldsymbol{\psi} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$,
along with any quantum observable
in the form of a self-adjoint matrix $\mathbf{A}$
whose spectral decomposition is $\mathbf{A} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \, \mathbf{\Pi}_{L_\lambda^{\mathbf{A}}}$.

The pair $(\psi, \mathbf{A})$ induces
a random variable $\mathbb{C}^n \ni \mathbf{x} \mapsto f^{\mathbf{A}}(\mathbf{x}) \in \mathbb{R} \cup \{*\}$
on the ortho-measurable space $(\mathbb{C}^n, \Sigma^{\mathbf{A}})$.

Its cumulative distribution function $\mathbb{R} \ni r \mapsto F_{\boldsymbol{\psi}}^{\mathbf{A}}(r) \in [0, 1]$
takes the form

$$F_{\boldsymbol{\psi}}^{\mathbf{A}}(r) = \mathbb{P}_{\boldsymbol{\psi}}^{\mathbf{A}}\left((f^{\mathbf{A}})^{-1}(-\infty, r]\right) = \sum_{\lambda \in s^{\mathbf{A}}} 1_{\lambda \leq r}(\lambda) \frac{\boldsymbol{\psi}^* \mathbf{\Pi}_{L_\lambda^{\mathbf{A}}} \boldsymbol{\psi}}{\boldsymbol{\psi}^* \boldsymbol{\psi}}$$

Because of the spectral decomposition of $\mathbf{A}$ and linearity,
the expectation of the induced random variable $f^{\mathbf{A}}$ is

$$\mathbb{E}_{\boldsymbol{\psi}} f^{\mathbf{A}} = \sum_{\lambda \in s^{\mathbf{A}}} \lambda \frac{\boldsymbol{\psi}^* \mathbf{\Pi}_{L_\lambda^{\mathbf{A}}} \boldsymbol{\psi}}{\boldsymbol{\psi}^* \boldsymbol{\psi}} = \frac{\boldsymbol{\psi}^* \mathbf{A} \boldsymbol{\psi}}{\boldsymbol{\psi}^* \boldsymbol{\psi}}$$

A Multi-Probability Space with Pure Quantum States

Given any wave vector $\boldsymbol{\psi} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, we can now use the family $(\mathbb{P}^D_{\boldsymbol{\psi}})_{D \in \mathcal{D}}$ of ortho-probability measures to extend

- our previous multi-measurable space $(\mathbb{C}^n, (\Sigma^D)_{D \in \mathcal{D}})$, with a complete family of ortho-algebras, one for each orthogonal decomposition $D \in \mathcal{D}$;
- into a multi-probability space $(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D_{\boldsymbol{\psi}})_{D \in \mathcal{D}})$, with a complete family of contextual probability spaces $(\mathbb{C}^n, \Sigma^D, \mathbb{P}^D_{\boldsymbol{\psi}})$, one for each orthogonal decomposition $D \in \mathcal{D}$.

Robert B. Griffiths (2002) *Consistent Quantum Theory* (Cambridge University Press).

Griffiths calls each of these contextual probability spaces a "single framework".

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A Contextual Multi-Measurable Space

A Contextual Multi-Probability Space

- Pre-Probability Wave Vectors as Pure Quantum States
- Density Matrices as Mixed Quantum States

Conclusion: Probabilistic Foundations of Quantum Mechanics?

Density Matrices and Their Spectral Decomposition

Definition

A self-adjoint matrix $\boldsymbol{\rho}$ is a density matrix or mixed state just in case there is an orthonormal basis $\mathcal{B}$ of $\mathbb{C}^n$ for which the spectral decomposition $\boldsymbol{\rho} = \sum_{\pi \in \mathbf{s}^{\boldsymbol{\rho}}} \pi \, \mathbf{\Pi}_{L_\pi}$ of $\boldsymbol{\rho}$ can be expressed as the probability mixture $\boldsymbol{\rho} = \sum_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} \, \mathbf{b} \, \mathbf{b}^*$ of orthogonal projections onto the one-dimensional spaces $[\mathbf{b}]$ spanned by the respective basis elements $\mathbf{b}$, where (by definition of probability mixture) one has:

1. $\pi_{\mathbf{b}} \geq 0$ for all $\mathbf{b} \in \mathcal{B}$;
2. $\sum_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} = 1$.

In $\mathbb{C}^n$ this definition is equivalent, but perhaps more transparent, than what has become standard in quantum theory.

It is evidently equivalent to regarding an $n \times n$ density matrix as an ortho-probability measure defined on an ortho-algebra.

Compounding Probabilities in a Multi-Probability Space

Given the decomposition $\boldsymbol{\rho} = \sum_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} \, \mathbf{b} \, \mathbf{b}^*$
of a density matrix or mixed state $\boldsymbol{\rho}$,
each vector $\mathbf{b}$ of the orthonormal basis $\mathcal{B}$
is a normalized wave vector or pure state that determines
a multi-probability space $\left(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D_{\mathbf{b}})_{D \in \mathcal{D}}\right)$ where,
for each orthogonal decomposition $D \in \mathcal{D}$ and each $E \in \Sigma^D$,
the fact that $\mathbf{b}^* \, \mathbf{b} = 1$ implies $\mathbb{P}^D_{\mathbf{b}}(E) = \mathbf{b}^* \, \Pi_E \mathbf{b}$.

The laws for compounding probabilities
imply that each mixed state $\boldsymbol{\rho} = \sum_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} \, \mathbf{b} \, \mathbf{b}^*$
gives rise to a multi-probability space $\left(\mathbb{C}^n, (\Sigma^D, \mathbb{P}^D_{\boldsymbol{\rho}})_{D \in \mathcal{D}}\right)$ where,
for each orthogonal decomposition $D \in \mathcal{D}$ and each $E \in \Sigma^D$,

$$\mathbb{P}^D_{\boldsymbol{\rho}}(E) = \sum\nolimits_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} \mathbb{P}^D_{\mathbf{b}}(E) = \sum\nolimits_{\mathbf{b} \in \mathcal{B}} \pi_{\mathbf{b}} \mathbf{b}^* \, \Pi_E \mathbf{b}$$

Outline

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Research Question: Can Quantum Nodes Be Reduced?

Claim (repeated)

*Under precise conditions still being investigated,
any quantum node in a quantum decision tree
can be reduced to a three-stage process
that compounds three different kinds of node:*

1. *a decision node (possibly trivial) that determines
a quantum experiment with a specified Boolean algebra
based on an orthogonal decomposition of* $\mathbb{C}^n$*;*
2. *a "quantum event" node, where a horse lottery
with subjective probabilities specified by a density operator
— or equivalently, by a Bayesian prior probability distribution
— determines a wave vector for the quantum experiment;*
3. *a "quantum chance" node where, given the wave vector,
a "quantum roulette lottery" determines the magnitude
of the relevant random "quantum observable",
with "objective" probabilities specified by Born's rule.*

Early Works by von Neumann, then Kolmogorov

John von Neumann:
(1927) "Mathematische Begründung der Quantenmechanik"
and "Wahrscheinlichkeitstheoretischer Aufbau
der Quantenmechanik"
*Nachrichten von der Gesellschaft der Wissenschaften
zu Göttingen (Mathematisch-Physikalische Klasse 1927)*:
1–57 and 245–272.

(1932) *Mathematische Grundlagen der Quantenmechanik*

Reaction (?) to Paul Dirac (1930)
The Principles of Quantum Mechanics (Oxford Univ. Press)

Andrey Nikolayevich Kolmogorov:
(1933) *Grundbegriffe der Wahrscheinlichkeitsrechnung*

Too late for von Neumann?

Does this still leave scope for the "mathematical foundations"
to be updated to the "probabilistic foundations"?

Envoi

Many thanks for your patience and attention!