Lorentzian polynomials on cones. Part II

Jonathan Leake

University of Waterloo

based on joint work with

Petter Brändén

KTH Royal Institute of Technology

Fields Institute, March 9, 2023

Motivation: Geometric inequalities

- Brunn-Minkowski inequality (1887). For convex bodies $K_1, K_2 \subset \mathbb{R}^n$,

$$\mathrm{Vol}(K_1 + K_2)^{1/n} \geq \mathrm{Vol}(K_1)^{1/n} + \mathrm{Vol}(K_2)^{1/n},$$

  where $K_1 + K_2 = \{x_1 + x_2 : x_1 \in K_1 \text{ and } x_2 \in K_2\}$.

- Minkowski. For convex bodies $K_1, \ldots, K_m$, and $x_1, \ldots, x_m > 0$,

$$\mathrm{Vol}(x_1 K_1 + \cdots + x_m K_m) = \sum_{i_1,\ldots,i_d} V(K_{i_1}, \ldots, K_{i_n}) x_{i_1} \cdots x_{i_n},$$

  where $V(K_1, \ldots, K_n) \geq 0$ are the mixed volumes.

- Alexandrov-Fenchel inequalities (1937).

$$V(K_1, K_2, \ldots, K_n)^2 \geq V(K_1, K_1, K_3 \ldots, K_n) \cdot V(K_2, K_2, K_3, \ldots, K_n)$$

Motivation: Elements of Hodge theory

- Let

$$A = \mathbb{R}[x_1, \dots, x_n]/I = \bigoplus_{k=0}^{d} A^k$$

is a graded $\mathbb{R}$-algebra.

- Suppose A^d is one-dimensional, and let

$$\deg : A^d \to \mathbb{R}$$

be a linear isomorphism.

- Suppose $\mathcal{K} \subset A^1$ is an open convex cone.

Kähler package

Desirable properties of A.

Poincaré duality (PD)

The bilinear map,

$$A^k \times A^{d-k} \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \deg(xy),$$

is nondegenerate.

Hard Lefschetz property (HL)

For each $0 \leq k \leq d/2$, and any $\ell_1, \ell_2, \ldots, \ell_{d-2k} \in \mathcal{K}$, the linear map

$$A^k \longrightarrow A^{d-k}, \quad x \longmapsto \ell_1 \ell_2 \cdots \ell_{d-2k} x,$$

is bijective.

Kähler package

Hodge-Riemann relations (HR)

For each $0 \leq k \leq d/2$, and any $\ell_0, \ell_1, \ldots, \ell_{d-2k} \in \mathcal{K}$, the bilinear map

$$A^k \times A^k \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto (-1)^k \deg(\ell_1 \ell_2 \cdots \ell_{d-2k} xy)$$

is positive definite on $\{x \in A^k : \ell_0 \ell_1 \cdots \ell_{d-2k} x = 0\}$.

Let $\ell_1, \ldots, \ell_d \in \mathcal{K}$.

(P) For $k = 0$, (HR) says $\deg(\ell_1 \ell_2 \cdots \ell_d) > 0$.

(AF) For $k = 1$, (HR) says

$$\deg(\ell_1 \ell_2 \ell_3 \cdots \ell_d)^2 \geq \deg(\ell_1 \ell_1 \ell_3 \cdots \ell_d) \deg(\ell_2 \ell_2 \ell_3 \cdots \ell_d).$$

(LC) In particular, the sequence $a_k = \deg(\ell_1^k \ell_2^{d-k})$ is log-concave

$$a_k^2 \geq a_{k-1} a_{k+1}, \quad 0 < k < d.$$

Examples

- Classical examples of Kähler package comes from compact Kähler manifolds and projective varieties,
- Polytopes (Stanley, McMullen),
- Chow rings of matroids (Adiprasito, Huh, Katz), and similar Chow rings.

Beyond Hodge theory

- Is there a common "geometry of polynomials" setting for these examples?
- The degree map defines a homogeneous degree d polynomial in $\mathbb{R}[t_1, \ldots, t_n]$:

$$\mathrm{vol}_A(t) = \frac{1}{d!} \deg \left(\left(\sum_{i=1}^{n} t_i x_i \right)^d \right). \qquad \text{(volume polynomial)}$$

- Let $\ell = a_1 x_1 + \cdots + a_n x_n \in A^1$, $v = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then

$$D_v \, \mathrm{vol}_A(t) = \sum_{i=1}^{n} a_i \partial_i \, \mathrm{vol}_A(t) = \frac{1}{(d-1)!} \deg \left(\ell \cdot \left(\sum_{i=1}^{n} t_i x_i \right)^{d-1} \right)$$

- Iterate: $D_{v_1} D_{v_2} \cdots D_{v_d} \, \mathrm{vol}_A(t) = \deg(\ell_1 \ell_2 \cdots \ell_d)$.

Lorentzian polynomials on cones

- Let $f \in \mathbb{R}[t_1, \ldots, t_n]$ be a homogeneous degree d polynomial.
- Let $\mathcal{K}$ be an open convex cone in $\mathbb{R}^n$.
- f is called $\mathcal{K}$-Lorentzian if for all $v_1, \ldots, v_d \in \mathcal{K}$,

(P) $D_{v_1} \cdots D_{v_d} f > 0$, and

(AF) $(D_{v_1} D_{v_2} \cdots D_{v_d} f)^2 \geq (D_{v_1} D_{v_1} \cdots D_{v_d} f)(D_{v_2} D_{v_2} \cdots D_{v_d} f)$

- Hence we get $\mathcal{K}$-Lorentzian polynomials from the examples from Hodge theory above.
- Example. The determinant $A \mapsto \det(A)$ is Lorentzian on the cone of positive definite matrices.
- Example. A hyperbolic polynomial (Petrovsky, Gårding) is Lorentzian on its hyperbolicity cone.
- There are $\mathcal{K}$-Lorentzian polynomials that do not come from any of the examples from Hodge-theory above.

Hereditary polynomials

- Let V be a finite set and $f \in \mathbb{R}[t_i : i \in V]$ a homogeneous polynomial of degree d.
- The lineality space of f is

$$\begin{aligned} L_f &= \{v \in \mathbb{R}^V : f(t+v) = f(t) \text{ for all } t \in \mathbb{R}^V\} \\ &= \{v \in \mathbb{R}^V : D_v f \equiv 0\}. \end{aligned}$$

- Define a simplicial complex on V by

$$\Delta_f = \{S \subseteq V : \partial^S f \not\equiv 0\}, \quad \partial^S = \prod_{i \in S} \frac{\partial}{\partial t_i}.$$

- f is hereditary if for each $S \in \Delta_f$ with $|S| = d - 1$,

$$\{(\ell_i)_{i \in S} : (\ell_i)_{i \in V} \in L_f\} = \mathbb{R}^S.$$

Hereditary polynomials

- Lemma. If f is hereditary and $S \in \Delta_f$, then

$$f^S(t) := \partial^S f\big|_{t_i=0, i\in S}$$

  is hereditary, with $\Delta_{f^S} = \mathrm{lk}_{\Delta_f}(S)$.
- Lemma. If f is hereditary, then Δ_f is pure of dimension $d-1$.
- Euler's formula then implies a recursive formula for hereditary polynomials:

$$d \cdot f(t) = \sum_{i \in V} t_i \cdot f^{\{i\}}(\pi_{\{i\}}(t)),$$

  where π_S is a linear projection for which $\pi_S(L_f) \subseteq L_{f^S}$.
- Corollary. Every hereditary polynomial is determined by its linear coefficients, given by $w(F) := f^F$ for all facets $F \in \Delta_f$.

Hereditary polynomials

- Corollary. f hereditary implies for all $S \in \Delta_f$ with $|S| = d-1$,

$$f^S(t) = \sum_{i \notin S} w(S \cup \{i\}) \cdot t_i$$

  and $f^S(t)$ is identically zero on $\pi_S(L_f)$.
- Converse is also true.
- Let Δ be a pure of dim. $d-1$, and let $L \subseteq \mathbb{R}^V$ be linear.
- (Δ, L) is hereditary if for each $S \in \Delta$ with $|S| = d-1$,

$$\{(\ell_i)_{i \in S} : (\ell_i)_{i \in V} \in L\} = \mathbb{R}^S.$$

- Lemma. Let (Δ, L) be hereditary. Function w on facets of Δ defines unique hereditary poly. with $\Delta_f = \Delta$ and $L \subseteq L_f$ iff

$$\sum_{i \notin S} w(S \cup \{i\}) \cdot t_i$$

  is identically zero on $\pi_S(L)$ for all $S \in \Delta$ with $|S| = d-1$.
- The function w is analogous to a Minkowski weight on a fan.

Hereditary Lorentzian polynomials

- For a hereditary polynomial f of degree d, there is a canonically defined open convex cone $\mathcal{K}_f$ in $\mathbb{R}^V$.
- $\mathcal{K}_f$ can be defined inductively via:
- if $d = 1$ then $\mathcal{K}_f := \{v \in \mathbb{R}^V : f(v) > 0\}$,
- if $d \geq 2$ then $\mathcal{K}_f$ is the set of all $v \in \mathbb{R}^V$ such that
  1. $v + \ell \in \mathbb{R}^V_{>0}$ for some $\ell \in L_f$, and
  2. $\pi_{\{i\}}(v) \in \mathcal{K}_{f^{\{i\}}}$ for all $i \in V$.
- f is called hereditary Lorentzian if f^S is $\mathcal{K}_{f^S}$-Lorentzian for all $S \in \Delta_f$ with $|S| \leq d - 1$.
- Δ_f is H-connected if for each $S \in \Delta_f$, $|S| \leq d - 3$, the graph

$$\Big\{\{i,j\} : S \cap \{i,j\} = \varnothing \text{ and } S \cup \{i,j\} \in \Delta_f\Big\}$$

is connected.

Hereditary Lorentzian polynomials

- Theorem (Brändén-L). Let f be a hereditary polynomial of degree d with $\mathcal{K}_f \neq \varnothing$. Then f is hereditary Lorentzian if and only if
  - (C) Δ_f is H-connected, and
  - (L) For each $S \in \Delta_f$ with $|S| = d - 2$, the Hessian of f^S has at most one positive eigenvalue.
- Example. Volume polynomials of matroids.
  - Implies the Heron-Rota-Welsh conjecture on the characteristic polynomial of a matroid.
- Example. Volume polynomials of simple polytopes.
  - Implies the Alexandrov-Fenchel inequalities for convex bodies.
- Example. Volume polynomials of Chow rings of fans.
  - Both the matroid (Adiprasito-Huh-Katz) and polytope cases (Stanley-McMullen) fit into this context.

Volume polynomial of a matroid

- Let $\mathcal{L}$ be the lattice of flats of a rank-$(d+1)$ matroid M on E, with set of loops K, and let $\underline{\mathcal{L}} = \mathcal{L} \setminus \{K, E\}$.
- The faces of the $(d-1)$-dim. order complex, $\Delta(\underline{\mathcal{L}})$, are $\{F_1 < F_2 < \cdots < F_k\}$, where $F_i \in \underline{\mathcal{L}}$ for all i.
- Define $L(\underline{\mathcal{L}}) \subseteq \mathbb{R}^{\underline{\mathcal{L}}}$ as subspace of all modular $(y_F)_{F \in \underline{\mathcal{L}}}$, i.e.

$$y_F = \sum_{i \in F \setminus K} c_i \quad \text{and} \quad \sum_{i \in E \setminus K} c_i = 0$$

  for some choice of $c_i \in \mathbb{R}$ for all $i \in E \setminus K$.
- $(\Delta(\underline{\mathcal{L}}), L(\underline{\mathcal{L}}))$ is hereditary.

Volume polynomial of a matroid

- For every facet $T \in \Delta(\mathcal{L})$, define $w(T) = 1$.
- For all $S \in \Delta(\mathcal{L})$ with $|S| = d - 1$,

$$\sum_{G \notin S} w(S \cup \{G\}) \cdot t_G = \sum_{F_i \prec G \prec F_{i+1}} t_G$$

is identically zero on $\pi_S(L(\mathcal{L}))$, since the rank-one flats of a matroid partition its non-loop elements.

- The volume polynomial of M is unique hereditary polynomial $f_{\mathcal{L}}$ defined by w, with $\Delta_{f_{\mathcal{L}}} = \Delta(\mathcal{L})$ and $L(\mathcal{L}) \subseteq L_{f_{\mathcal{L}}}$.
- Note that $\Delta_{f_{\mathcal{L}}^S} = \mathrm{lk}_{\Delta(\mathcal{L})}(S) = \prod_i \Delta([F_i, F_{i+1}])$.
- Uniqueness then implies

$$f_{\mathcal{L}}^S(t) = \prod_i f_{[F_i, F_{i+1}]}(t).$$

Volume polynomial of a matroid

- The canonical cone $\mathcal{K}_{f_\mathcal{L}}$ is non-empty because it contains the set of strictly submodular $(x_S)_{K \subset S \subset E}$, i.e.

$$x_S + x_T > x_{S \cup T} + x_{S \cap T} \quad \text{with} \quad x_K = x_E = 0$$

  for uncomparable S, T.
- H-connectivity of $\Delta(\mathcal{L})$ follows from semimodularity of $\mathcal{L}$.
- To prove $f_\mathcal{L}$ is hereditary Lorentzian it remains to consider $f_\mathcal{L}^S(t)$ for $S \in \Delta(\mathcal{L})$ with $|S| = d - 2$.
- Either such a quadratic is product of two linear polynomials, or
- it is the volume polynomial of a matroid of rank 3:

$$\left(\sum_{K \prec F} t_F \right)^2 - \sum_{G \prec E} \left(t_G - \sum_{K \prec F \prec G} t_F \right)^2,$$

  which has exactly one positive eigenvalue.

Chow rings of simplicial fans

- Let Δ be a simplicial complex of dimension $d-1$ on V.
- Let $\Sigma = \{C_S\}_{S\in\Delta}$ be a collection of $|S|$-dimensional polyhedral cones such that
  - Each face of C_S is a cone in Σ, and
  - $C_S \cap C_T = C_{S\cap T}$.
- Σ is called a simplicial fan.
- Let ρ_i, $i \in V$, be specified vectors of the rays $C_{\{i\}}$.
- Let $L = L(\Sigma) = \{(\lambda(\rho_i))_{i\in V} : \lambda \in \left(\mathbb{R}^V\right)^*\}$.
- (Δ, L) is hereditary.

Chow rings of simplicial fans

- Define two ideals in $\mathbb{R}[x_i : i \in V]$:
  - $I(\Delta)$ is generated by the monomials $\prod_{i \in T} x_i, \quad T \notin \Delta.$
  - $J(L)$ is generated by the linear forms $\sum_{i \in V} \ell_i x_i, \quad (\ell_i)_{i \in V} \in L.$
- The graded ring

$$A(\Sigma) = \bigoplus_{k=0}^{d} A^k(\Sigma) := \mathbb{R}[x_i : i \in V] \big/ (I(\Delta) + J(L))$$

  is the Chow ring of Σ.
- Important examples of Chow rings that satisfy the Kähler package are
  - The normal fan of a simple polytope (Stanley, McMullen).
  - The Chow ring of a matroid (Adiprasito, Huh and Katz), and related Chow rings.

Volume polynomials of simplicial fans

- Given any $\alpha \in A^k(\Sigma)^*$,

$$f_\alpha(t) = \frac{1}{k!}\alpha\Big(\Big(\sum_{i \in V} t_i x_i\Big)^k\Big)$$

  is a hereditary polynomial.
- When $A^d(\Sigma)$ is one-dimensional, then the given hereditary polynomial is called the volume polynomial of Σ.
- Our main theorem implies a characterization of α for which f_α is hereditary Lorentzian
- Corollary. Characterization of $A(\Sigma)$ satisfying (P) and (AF), the Hodge-Riemann relations of degree 0 and 1.
  - See also independent work of Dustin Ross.

Edge subdivisions of simplicial fans

- Fans have a natural notion of stellar subdivision:
- Add a new cone $C_{\{0\}}$ with ray ρ_0 in the relative interior of a cone C in Σ, and break C into many cones incident on $C_{\{0\}}$.
- If $C \sim \{\rho_1, \rho_2\}$ is two-dimensional, call it an edge subdivision.
- The support of a fan Σ is defined as $\bigcup_{C \in \Sigma} C$
- Stellar subdivisions of fans preserve the support of the fan, and
- whether or not a simplicial fan satisfies the Kähler package depends only on the support of the fan (Ardila-Denham-Huh)

Edge subdivisions of hereditary polynomials

- How does edge subdivision of $\{\rho_1, \rho_2\}$ act on Δ and L?
  - Get Δ_{12} by replacing $\{1,2\}$ by $\{1,0\}$ or $\{0,2\}$ in faces of Δ,
  - If $\rho_0 = c_1\rho_1 + c_2\rho_2$ for $c_1, c_2 > 0$ then

$$L_{12} = \{(\ell_0, \ell) \in \mathbb{R}^{\{0\}\cup V} : \ell \in L,\ \ell_0 = c_1\ell_2 + c_1\ell_2\}.$$

- Let (Δ, L) be hereditary, and let $\mathcal{P}^d(\Delta, L)$ be the degree d hereditary polynomials f such that $\Delta_f \subseteq \Delta$ and $L \subseteq L_f$.
- Proposition. Suppose $\{1,2\} \in \Delta$ and fix $c_1, c_2 > 0$. There is an injective linear map

$$\mathsf{sub}_{12} : \mathcal{P}^d(\Delta, L) \to \mathcal{P}^d(\Delta_{12}, L_{12})$$

which generalizes the map between the volume polynomials of fans. If $\dim(\Delta) = d - 1$ then sub_{12} is a bijection.

Edge subdivisions of hereditary polynomials

- For hereditary polynomials f, g, write $f \sim g$ if f and g are connected by a sequence of sub_{ij} or sub_{ij}^{-1} operations.
- Theorem (Brändén-L). If $f \sim g$ and $\mathcal{K}_f$ and $\mathcal{K}_g$ are non-empty, then f is hereditary Lorentzian if and only if g is hereditary Lorentzian.
- Corollary. Applies to volume polynomials of simplicial fans which have the same support.

Volume polynomials of simple polytopes

- Let $P \subset \mathcal{E}$ be a d-dim. simple polytope with facets $P_1, \ldots, P_n$ and associated facet unit (outward) normal vectors $\zeta_1, \ldots, \zeta_n$.
- Given Q with the same facet normals, define support numbers

$$t_i(Q) = \max_{q \in Q} \langle \zeta_i, q \rangle \quad \text{for} \quad 1 \leq i \leq n.$$

- The set of all such $t \in \mathbb{R}^n$ forms an open convex cone $\mathcal{K}_P$.
- There is volume polynomial f_P such that $f_P(t(Q)) = \mathrm{vol}(Q)$.
- Let Δ_P be the simplicial complex associated to the normal fan of P, and let L_P be defined via

$$L_P = \{(\langle v, \zeta_i \rangle)_{i=1}^n : v \in \mathcal{E}\}.$$

- (Δ_P, L_P) is hereditary and $f_P \in \mathcal{P}^d(\Delta_P, L_P)$.

Volume polynomials of simple polytopes

- Note that $\partial_i f_P(t(Q)) = \mathrm{vol}(Q_i)$, where Q_i is the facet of Q.
- This implies (see e.g. Schneider)

$$\partial_i f_P(t) = f_{P_i}\left(\left(\frac{t_j - t_i\cos(\theta_{ij})}{\sin(\theta_{ij})}\right)_j\right)$$

  where θ_{ij} is the angle between ζ_i and ζ_j.
- To prove that f_P is hereditary Lorentzian:
  - Δ_P is H-connected since the boundary of P is connected.
  - Need to show the Hessian of f_P has at most one positive eigenvalue, for any convex polygon P.
- If P is a triangle (simplex), then $f_P(t) = (a_1t_1 + a_2t_2 + a_3t_3)^2$ and $\mathcal{P}^2(\Delta_P, L_P)$ is one-dimensional since $\dim(L_P) = 2$.
- Apply edge subdivisions (vertex truncations) to obtain f_P for any convex polygon P; theorem implies hereditary Lorentzian.