The Universal Tutte Polynomial

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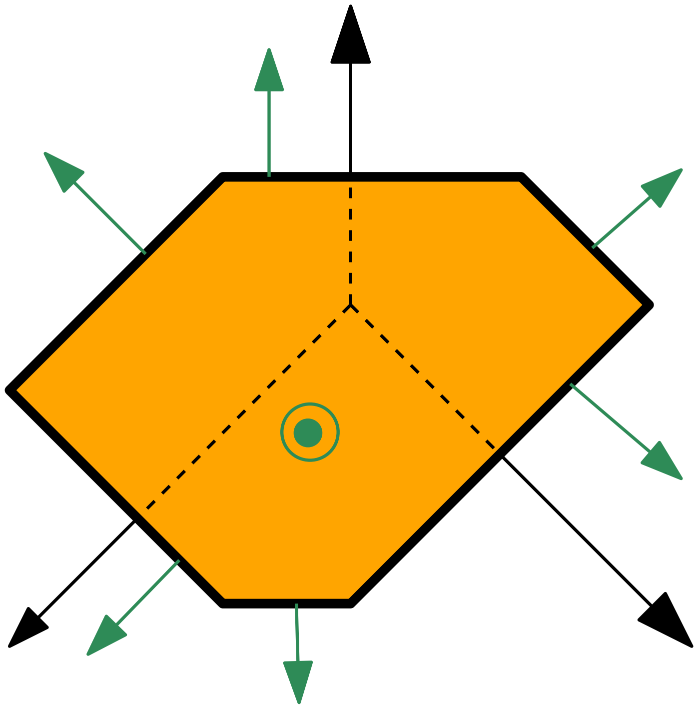
- Joint work with -

Tamás Kálmán (Tokyo IT) & Alex Postnikov (MIT)

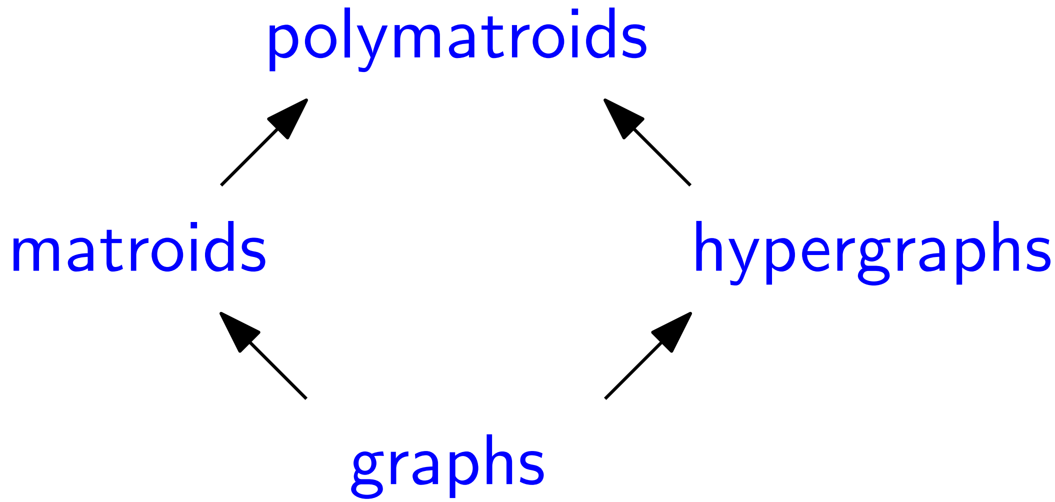


Fields Institute, October 2022

Outline/Motivation

1. Generalizing the Tutte polynomial to hypergraphs and polymatroids

$\mathbf{T}_P(x, y)$



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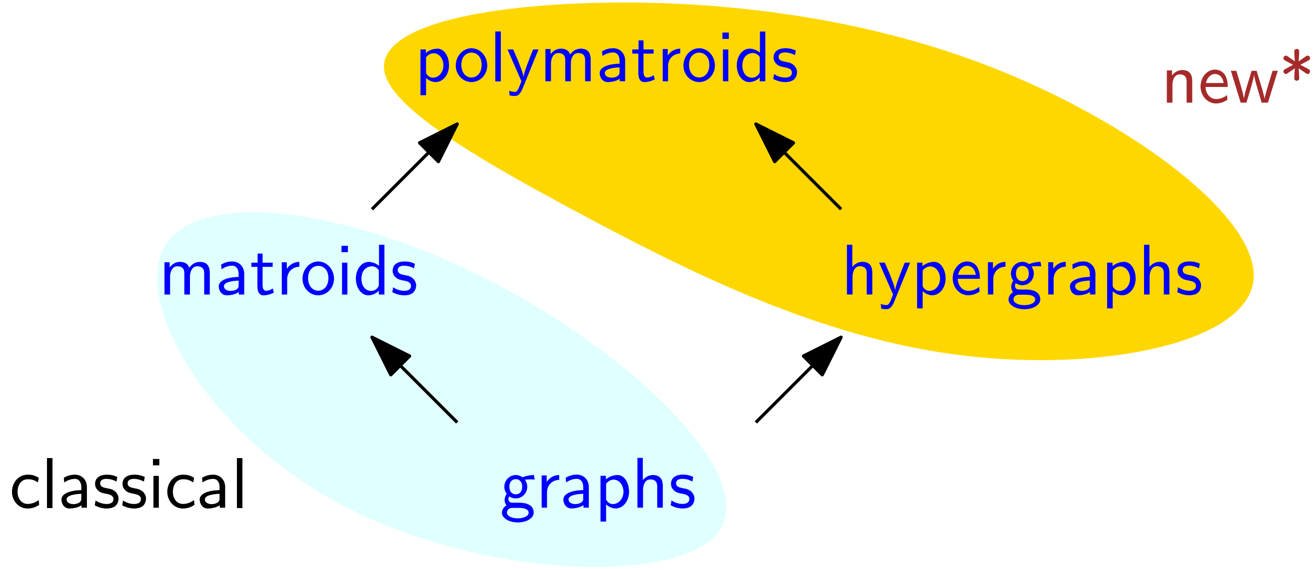

*completing results by Kálmán, Kálmán-Postnikov, and Cameron-Fink

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1. Generalizing the Tutte polynomial to hypergraphs and polymatroids

$\rightarrow$ extending known results from matroids to polymatroids,
$\rightarrow$ reflecting back on classical setting (e.g. Brilawsky identities),
$\rightarrow$ hypergraph invariants, knot invariants.

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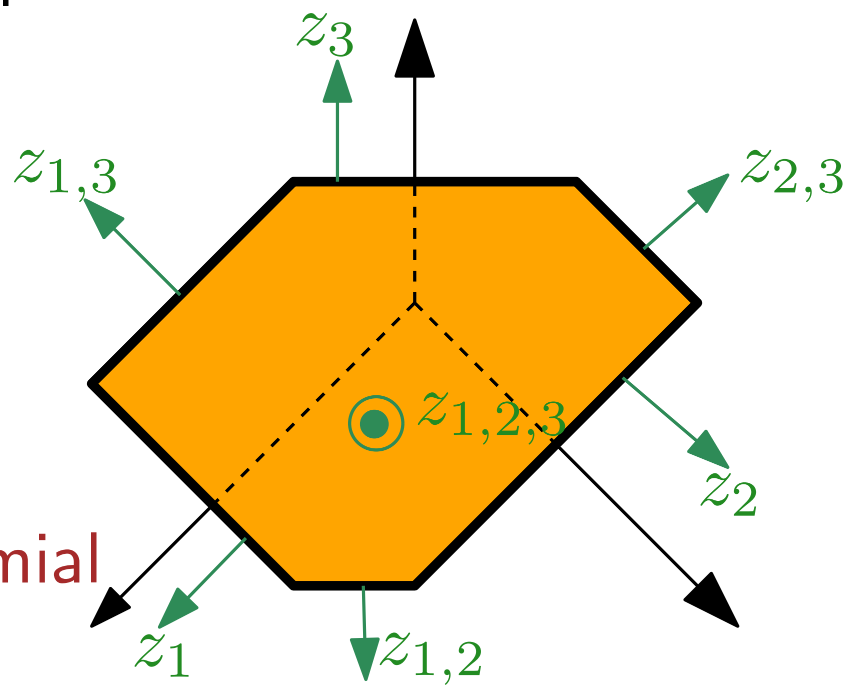
2. Universal Tutte polynomial

$\mathbf{T}_n(x, y; \mathbf{z})$



$\mathbf{T}_n$ parametrizes Tutte polynomial of polymatroids on $[n]$

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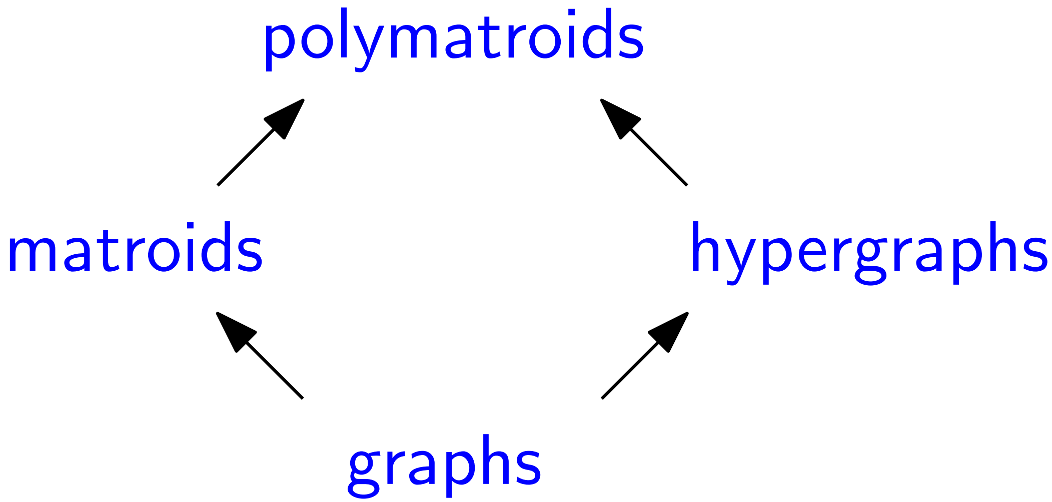
$\to$ extending known results from matroids to polymatroids,
$\to$ reflecting back on classical setting (e.g. Brilawsky identities),
$\to$ hypergraph invariants, knot invariants.

2. Universal Tutte polynomial

$\to$ coeffs of Tutte polynomials are polynomial in rank function,
$\to$ explicit expression of $\mathbf{T}_n$,
$\to$ connection with Postnikov's multi-Ehrhart polynomial of generalized permutahedra.

Background on polymatroids



Matroids

Def 1. A **matroid** on a set E is a set $M \subseteq 2^E$ of **bases** satisfying:

Exchange Axiom: $\forall A, B \in M,\ \forall i \in A \setminus B,$
$\exists j \in B \setminus A$ such that $A \cup \{j\} \setminus \{i\} \in M$ and $B \cup \{i\} \setminus \{j\} \in M.$

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Example: $E = \{1, 2, 3, 4\}$

$$\begin{aligned} M \quad = \ \{ & \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \\ & \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\} \} \end{aligned}$$

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Def 2. (**Base polytope**) A **matroid** on E is a polytope in $\mathbb{R}^E$ vertices in $\{0,1\}^E$ and edges in $\{\mathbf{e}_i - \mathbf{e}_j,\ i,j \in E\}$.

(notation: $\{\mathbf{e}_i,\ i \in E\}$ denotes the cannonical basis of $\mathbb{R}^E$)

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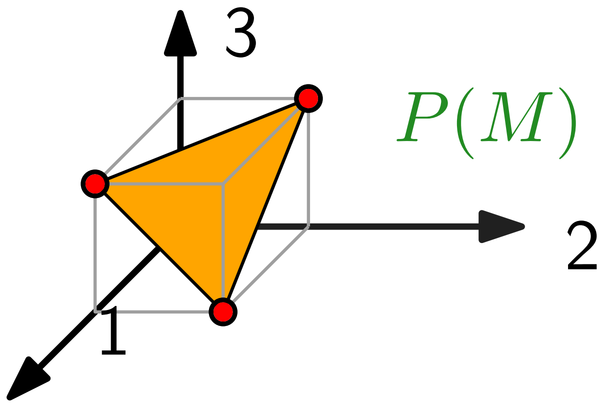
$M = \{\{1,2\}, \{2,3\}, \{1,3\}\}$



$P(M) = \text{conv}(M)$

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Def 3. (**Rank function**) A **matroid** on E is a polytope in $\mathbb{R}^E$, with faces of the form $\sum_{i \in S} x_i \leq f(S)$, and $\sum_{i \in E} x_i = f(E)$, where $f : 2^E \to \mathbb{N}$ is a **submodular function** such that $f(\{i\}) \leq 1$.

$$\forall S, T, \quad f(S) + f(T) \geq f(S \cup T) + f(S \cap T), \text{ with } f(\emptyset) = 0.$$

Rank function= unique submodular function f defining the facets.

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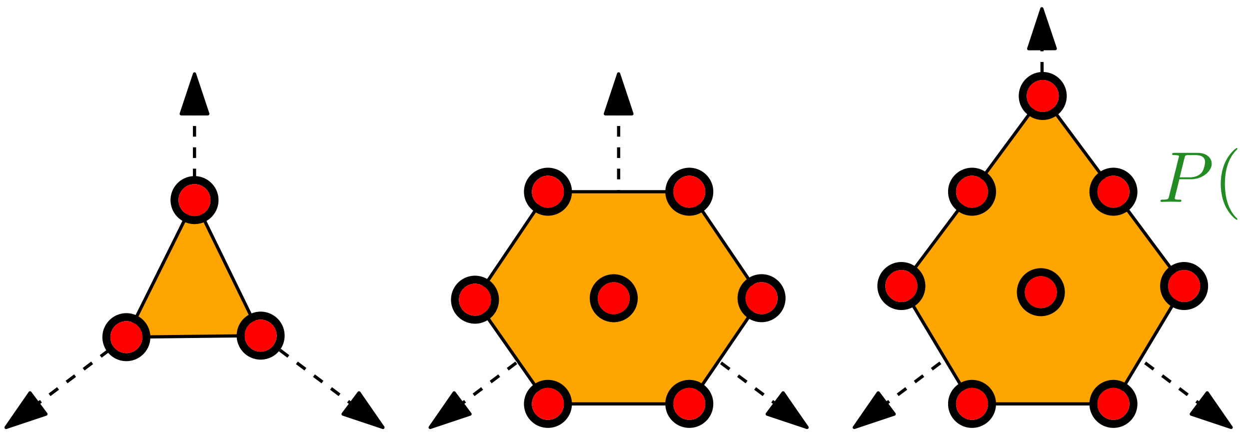

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"**generalized permutahedra**"

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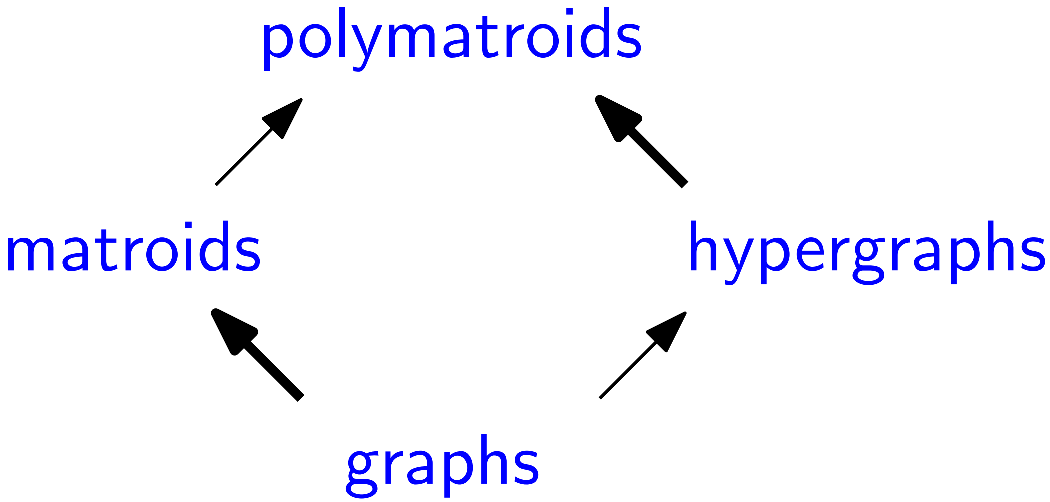

polymatroids
matroids
hypergraphs
graphs

Matroids from graphs

Prop. For any connected graph $G = (V, E)$,

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is a matroid on E.

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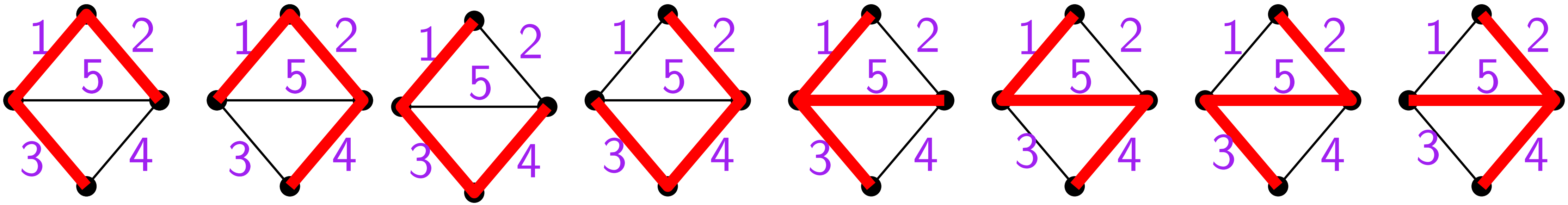
Example.

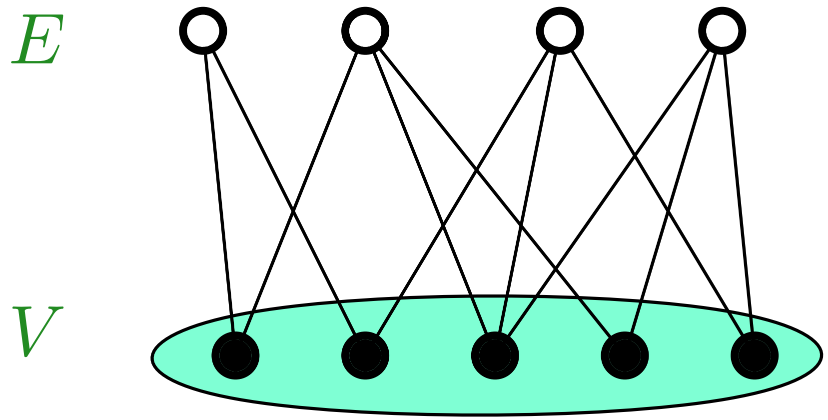


$$M_G = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,5\},\{2,4,5\}\}$$

Polymatroids from hypergraphs

Def: A **hypergraph** on a set V, is a multiset E of subsets of V.

Example:



Rk: Graph $=$ hypergraph where every hyperedge $e \in E$ has size 2.

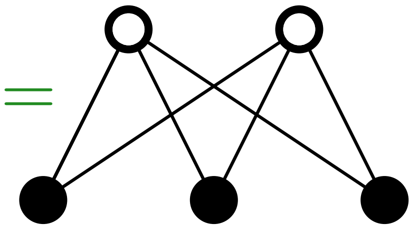
Polymatroids from hypergraphs

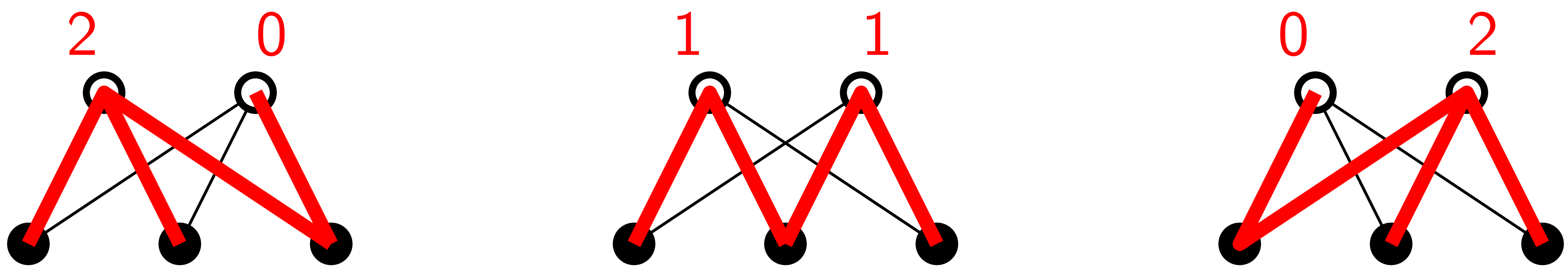
Def: Let $H = (V, E)$ be a hypergraph.
Let B_H be the corresponding bipartite graph.

A **spanning hypertree** of H is a point $\mathbf{a} \in \mathbb{N}^E$ for which there exists a spanning tree T of B_H such that

$$\forall i \in E, \quad a_i = \deg_T(i) - 1.$$

Example:

$H =$  has 3 hypertrees: $(2, 0), (1, 1), (0, 2)$



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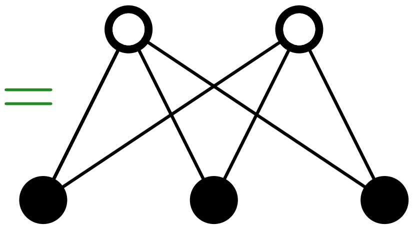
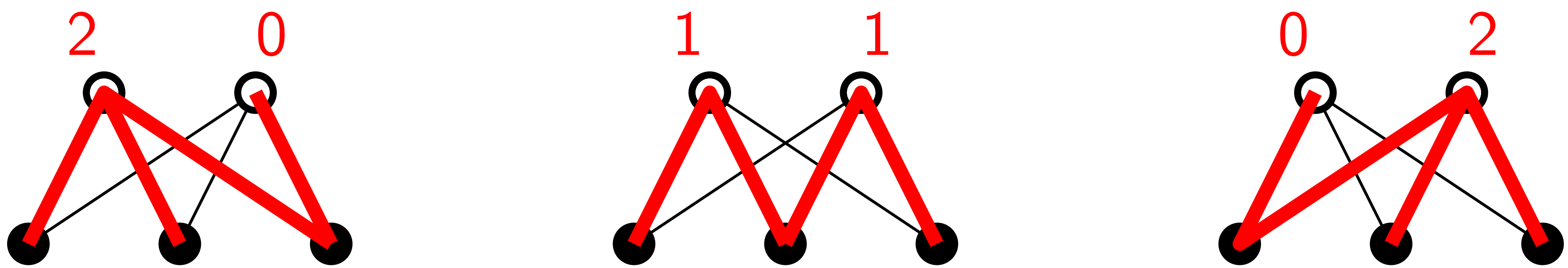
$H =$  has 3 hypertrees: $(2,0), (1,1), (0,2)$

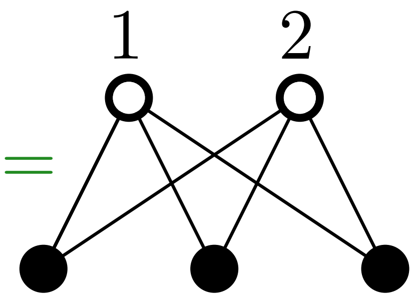


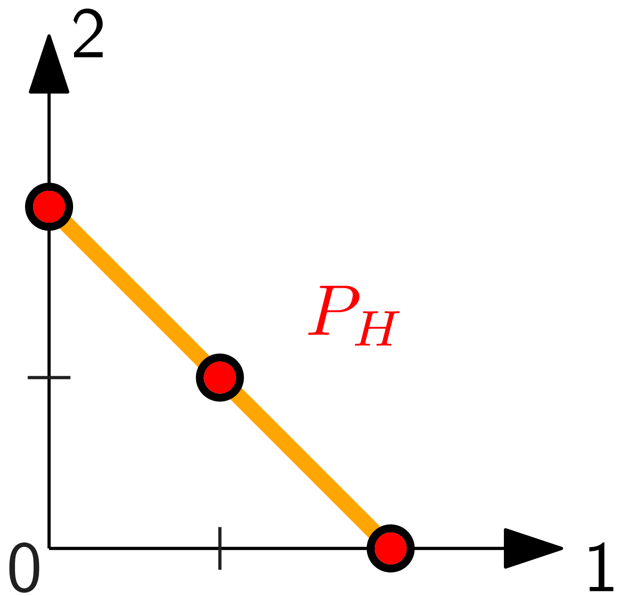
Remark: If a hypergraph H corresponds to a graph G, then the spanning hypertrees of H are in bijection with the spanning trees of G.

Polymatroids from hypergraphs

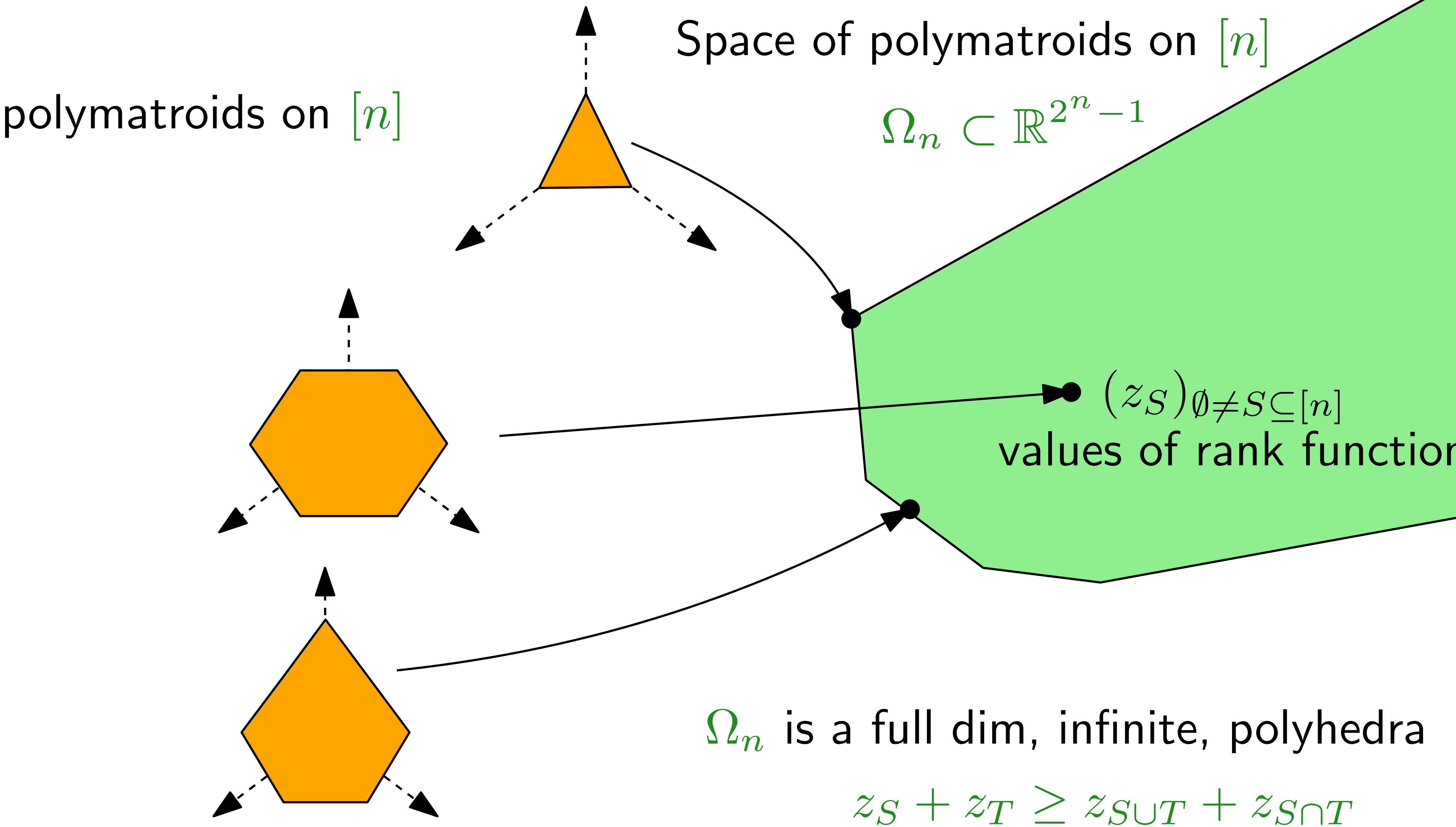
Prop: For any hypergraph $H = (V, E)$, the set of spanning hypertrees of H forms a polymatroid P_H on E.

Example:

$H =$ 



The space of polymatroids



{hypergraphs} contains a full dim, infinite, cone

Tutte polynomial of polymatroids

Tutte polynomial of matroids

Def: For a matroid M on E,

$$T_M(x,y) = \sum_{S \subseteq E} (x-1)^{\mathrm{cork}(S)} (y-1)^{\mathrm{null}(S)},$$

where

$$\begin{aligned} \mathrm{cork}(S) &= \text{ \# elements to add in order to contain a basis,} \\ \mathrm{null}(S) &= \text{ \# elements to delete in order to be contained in a basis.} \end{aligned}$$

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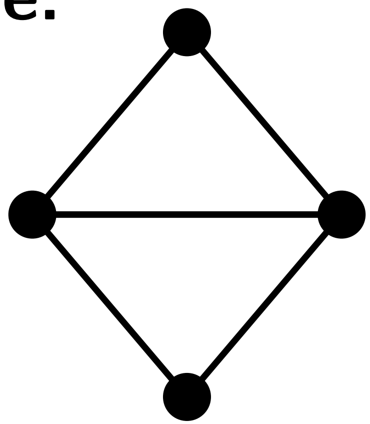
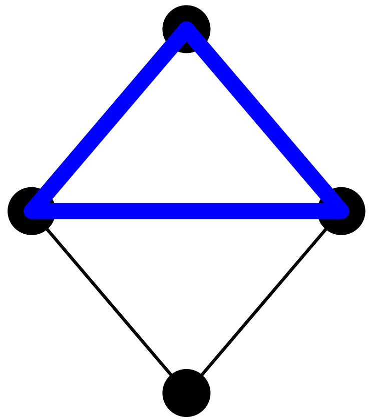
Example:

G



S



S contributes $(x-1)(y-1)$

$$T_G(x,y) = x^3 + 2x^2 + 2xy + y^2 + x + y.$$

Tutte polynomial of matroids

The Tutte poly $T_G(x, y)$ of a graph G captures a lot of information:

spanning trees, # forests, # connected subgraphs,
acyclic orientations, # totally cyclic orientations,
Chromatic polynomial, Potts polynomial,
G-parking functions by degree, Reliability polynomial...

Tutte polynomial of matroids

Thm: The Tutte polynomial is universal among invariants satysfying linear **deletion-contraction** formulas:

$$\forall i \in E \text{ neither loop nor coloop}, \quad X_M = \alpha\, X_{M\backslash i} + \beta\, X_{M/i}$$

$$\forall i \in E \text{ loop}, \quad X_M = \gamma\, X_{M\backslash i}$$

$$\forall i \in E \text{ coloop}, \quad X_M = \delta\, X_{M/i}$$

(The Tutte polynomial corresponds to $\alpha = \beta = 1$, $\gamma = y$, $\delta = x$)

Tutte polynomial of matroids

Thm [Tutte/Crapo] For any total order $\prec$ on E,

$$T_M(x,y) = \sum_{A \text{ basis}} x^{|\mathrm{IA}(A)|} y^{|\mathrm{EA}(A)|},$$

$$\begin{aligned} \mathrm{IA}(A) &= \{i \in A \mid \nexists j \prec i \text{ such that } A - i + j \text{ is a basis}\} \\ \mathrm{EA}(A) &= \{i \notin A \mid \nexists j \prec i \text{ such that } A + i - j \text{ is a basis}\} \end{aligned}$$

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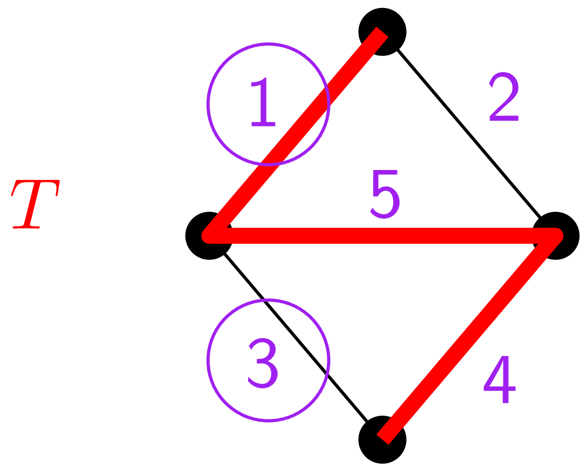

$\mathsf{IA}(T) = \{1\}$
$\mathsf{EA}(T) = \{3\}$

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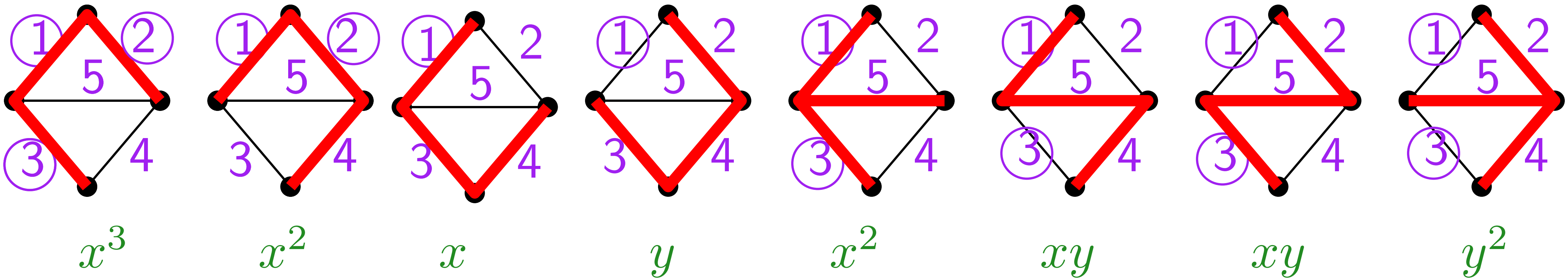
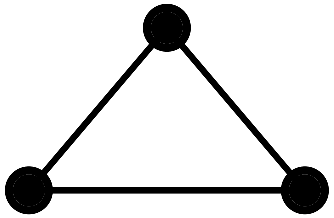


$$T_G(x,y) = x^3 + 2x^2 + 2xy + y^2 + x + y.$$

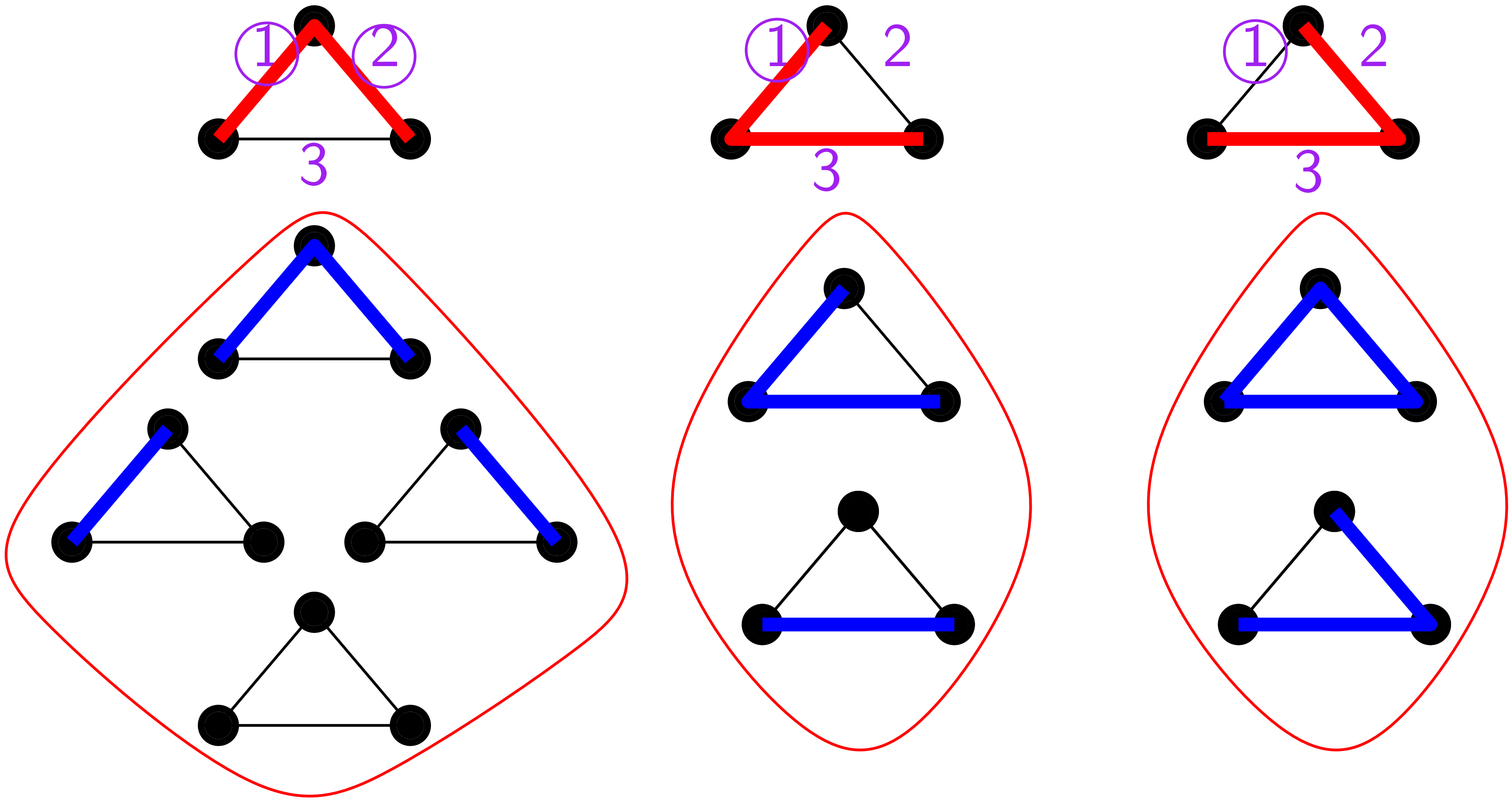
Relation between the two expressions of $T_M(x,y)$?

Example:

M

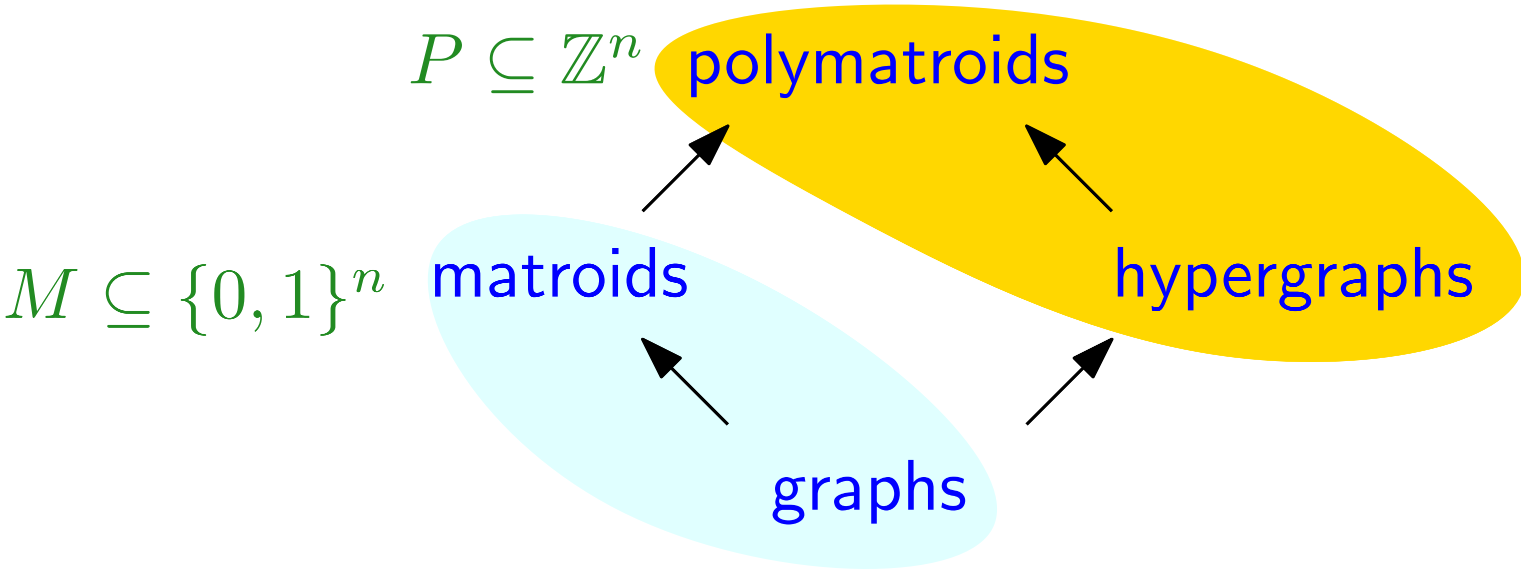


"Crapo's interval partition"



$$T_M(x,y) = x^2 + x + y.$$

Tutte polynomial of polymatroids?



Tutte polynomial of polymatroids?

Tentative definition: Let $P \subseteq \mathbb{Z}^n$ be a polymatroid.

For $\mathbf{a} \in P$, let

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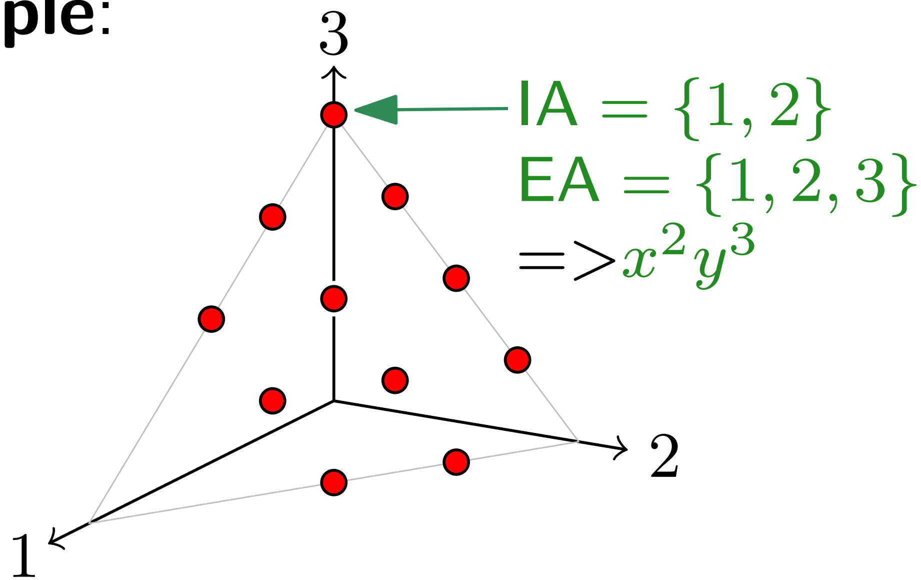

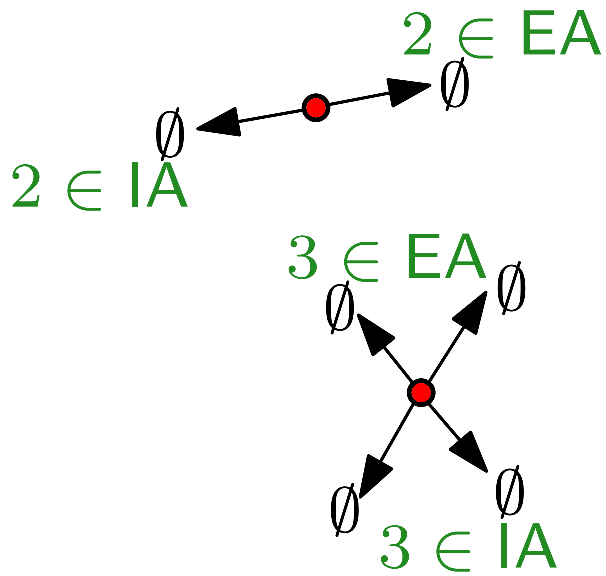

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However $T_P(x, 1)$ and $T_P(1, y)$ are invariant under reordering of $[n]$.
[Kalman 13, Kalman & Postnikov 17]

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$$\mathsf{EA}(\mathbf{a}) \overset{?}{=} \{i \in [n] \mid \nexists j < i \text{ such that } \mathbf{a} + \mathbf{e}_i - \mathbf{e}_j \in P\}.$$

$$T_P(x,y) \overset{?}{:=} \sum_{\mathbf{a} \text{ basis}} x^{|\mathsf{IA}(\mathbf{a})|} y^{|\mathsf{EA}(\mathbf{a})|}.$$

Does not work! Not invariant under reordering of $[n]$.

However $T_P(x,1)$ and $T_P(1,y)$ are invariant under reordering of $[n]$.
[Kalman 13, Kalman & Postnikov 17]

$\simeq$ "Ehrhart polynomial" $|(P + q\Delta) \cap \mathbb{Z}^n|$

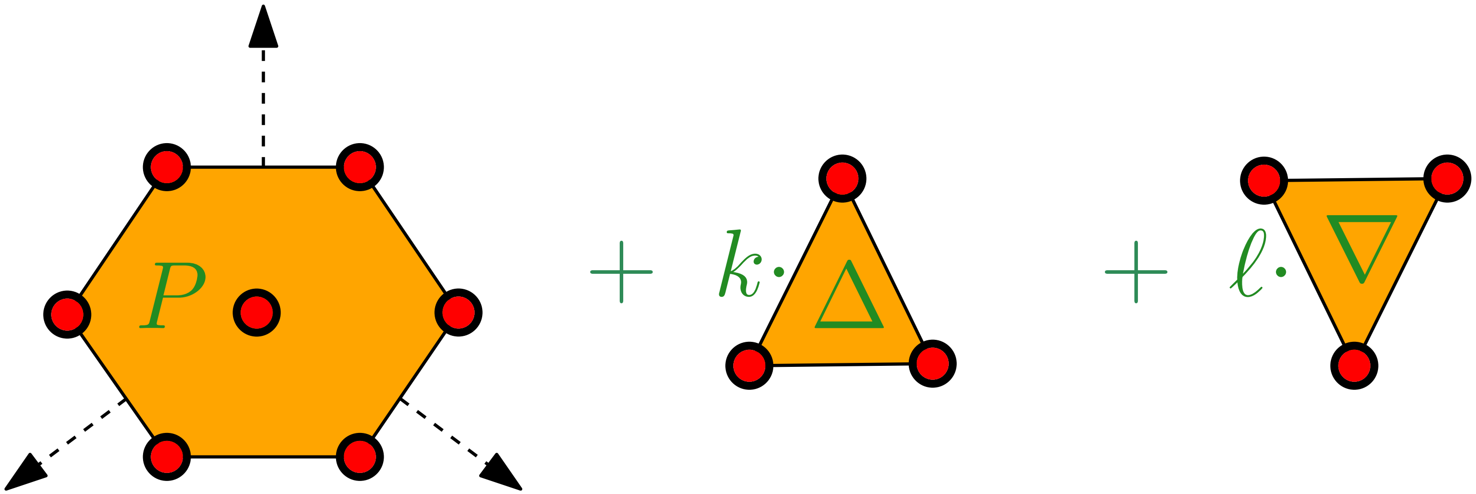
Cameron & Fink's fix

Def: The **Cameron-Fink invariant** for a polymatroid $P \subseteq \mathbb{Z}^n$ is the unique polynomial $Q_P(x,y)$ such that $\forall k, \ell \in \mathbb{Z}_{\geq 0}$,

$$Q_P(k,\ell) = \left|(P + k\,\nabla + \ell\,\Delta) \cap \mathbb{Z}^n\right|,$$

where $\Delta = \text{conv}(\mathbf{e}_i,\ i \in [n])$ and $\nabla = \text{conv}(-\mathbf{e}_i,\ i \in [n])$.

Example:



Prop [Cameron-Fink]: For a matroid M, $Q_M(x,y) \simeq T_{P(M)}(x,y)$

same information

Tutte polynomial of polymatroids.

Definition [BKP]: Let $P \subseteq \mathbb{Z}^n$ be a polymatroid.
For $\mathbf{a} \in P$

$$\begin{aligned} \mathsf{IA}(\mathbf{a}) &= \{i \in [n] \mid \nexists j < i \text{ such that } \mathbf{a} - \mathbf{e}_i + \mathbf{e}_j \in P\} \\ \mathsf{EA}(\mathbf{a}) &= \{i \in [n] \mid \nexists j < i \text{ such that } \mathbf{a} + \mathbf{e}_i - \mathbf{e}_j \in P\} \end{aligned}$$

$$\mathbf{T}_P(x,y) = \sum_{\mathbf{a} \text{ basis}} x^{|\mathsf{IA}(\mathbf{a}) \setminus \mathsf{EA}(\mathbf{a})|} \, y^{|\mathsf{EA}(\mathbf{a}) \setminus \mathsf{IA}(\mathbf{a})|} \, (x+y-1)^{|\mathsf{IA}(\mathbf{a}) \cap \mathsf{EA}(\mathbf{a})|}.$$

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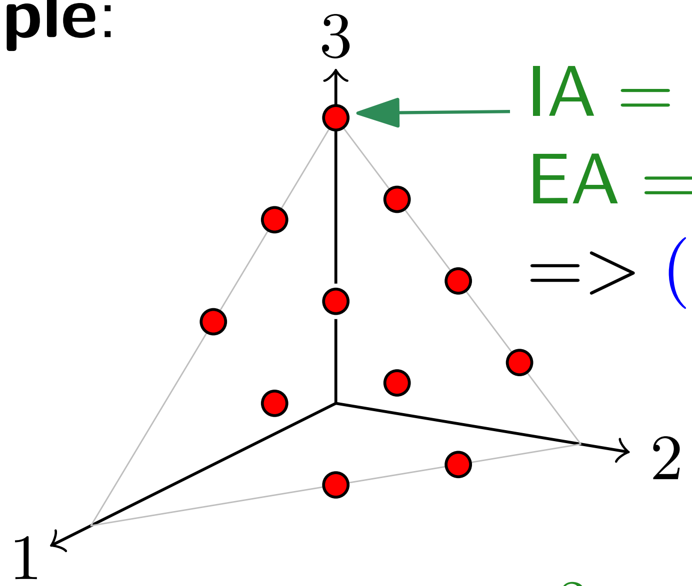
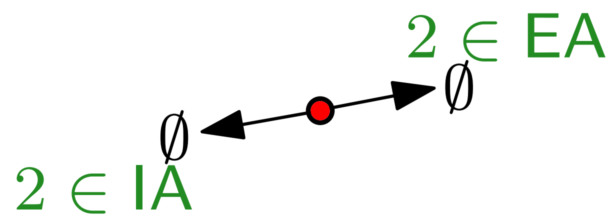
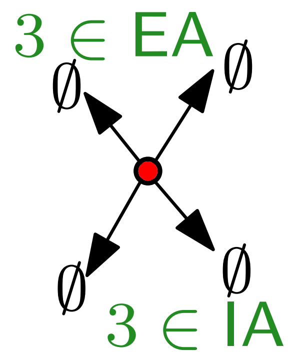


$\mathsf{IA} = \{1,2\}$

$\mathsf{EA} = \{1,2,3\}$

$\Rightarrow (x+y-1)^2 y$





$$\mathbf{T}_P(x,y) = (x+y-1)(x^2 + 2xy + y^2 + 2y + 3x + 2y + 2)$$

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Moreover, for any matroid M of rank d on $E = [n]$,

$$\mathbf{T}_{P(M)}(x,y) = x^{n-d}\, y^d\, T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right).$$

Tutte polynomial of polymatroids.

$$\mathbf{T}_P(x,y) = \sum_{\mathbf{a}\in\mathbb{Z}^b} ???$$

Interval partition?

Tutte polynomial of polymatroids.

Thm [BKP] For any polymatroid $P \subseteq \mathbb{Z}^n$,

$$\mathbf{T}_P\left(\frac{1}{1-u}, \frac{1}{1-v}\right) = \sum_{\mathbf{c}\in\mathbb{Z}^n} u^{\mathrm{cork}(\mathbf{c})} v^{\mathrm{null}(\mathbf{c})},$$

where $\mathrm{cork}(\mathbf{c}) = \min(\,|\mathbf{b}| \;\mid\; \mathbf{c}+\mathbf{b} \geq \mathbf{a} \in P)$,
$\mathrm{null}(\mathbf{c}) = \min(\,|\mathbf{b}| \;\mid\; \mathbf{c}-\mathbf{b} \leq \mathbf{a} \in P)$.

Tutte polynomial of polymatroids.

Relation: Crapo-type partition.

Let $P \subseteq \mathbb{Z}^n$ be a polynomatroid. For $\mathbf{a} \in P$ we define the cone

$$C(\mathbf{a}) = \mathbf{a} + \sum_{i \in \mathsf{IA}(\mathbf{a}) \setminus \mathsf{EA}(\mathbf{a})} \mathbb{Z}_{\leq 0}\, \mathbf{e}_i + \sum_{i \in \mathsf{EA}(\mathbf{a}) \setminus \mathsf{IA}(\mathbf{a})} \mathbb{Z}_{\geq 0}\, \mathbf{e}_i + \sum_{i \in \mathsf{IA}(\mathbf{a}) \cap \mathsf{EA}(\mathbf{a})} \mathbb{Z}\, \mathbf{e}_i.$$

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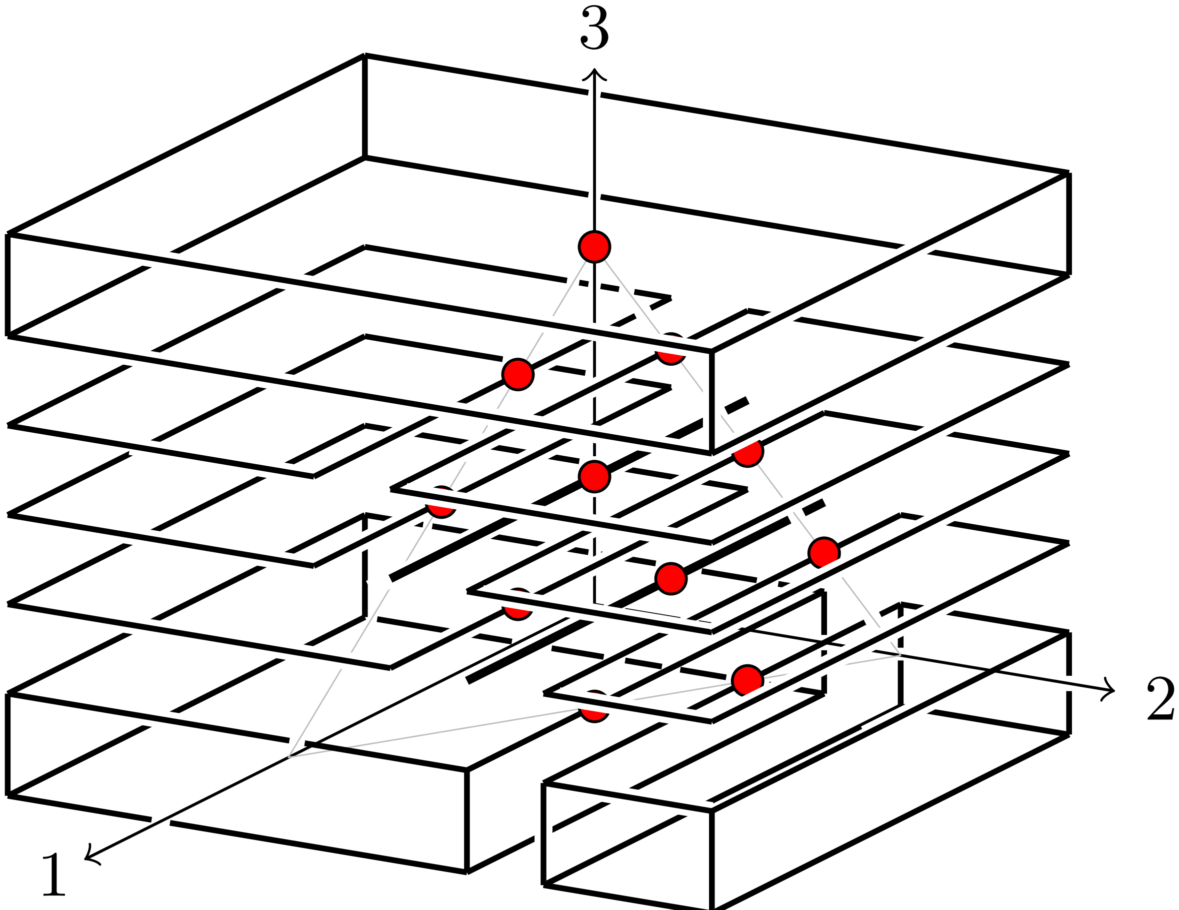
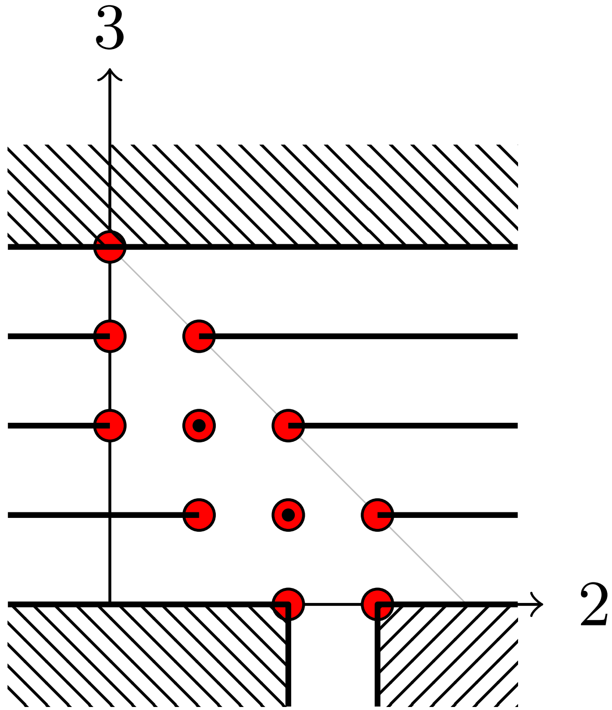
Example:





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Lemma [BKP] For any polymatroid $P \subseteq \mathbb{Z}^n$,

$$\biguplus_{\mathbf{a} \in P} C(\mathbf{a}) = \mathbb{Z}^n.$$

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Moreover, for all basis $\mathbf{a} \in P$,

$$\sum_{\mathbf{c} \in C(\mathbf{a})} u^{\mathrm{cork}(\mathbf{c})} v^{\mathrm{null}(\mathbf{c})} = \left(\frac{1}{1-u}\right)^{|\mathsf{IA}(\mathbf{a}) \setminus \mathsf{EA}(\mathbf{a})|} \left(\frac{1}{1-v}\right)^{|\mathsf{EA}(\mathbf{a}) \setminus \mathsf{IA}(\mathbf{a})|} \left(\frac{1}{1-u} + \frac{1}{1-v} - 1\right)^{|\mathsf{IA}(\mathbf{a}) \cap \mathsf{EA}(\mathbf{a})|}.$$

□

Relation with Cameron-Fink invariant

Prop [BKP]:

$$Q_P(x,y) = \sum_{i,j} c_{i,j} \binom{x}{i} \binom{y}{j},$$

where $c_{i,j} = [x^i y^j] \frac{\mathbf{T}_P(x+1, y+1)}{x+y+1}$.

Some properties of polymatroid Tutte polynomial

Prop. $P \subset \mathbb{R}^n$.

- $\mathbf{T}_P(x, y)$ is **invariant** under translation of P, and under permutation $[n]$.

- **Duality**: $\mathbf{T}_{-P}(x, y) = \mathbf{T}_P(y, x)$.

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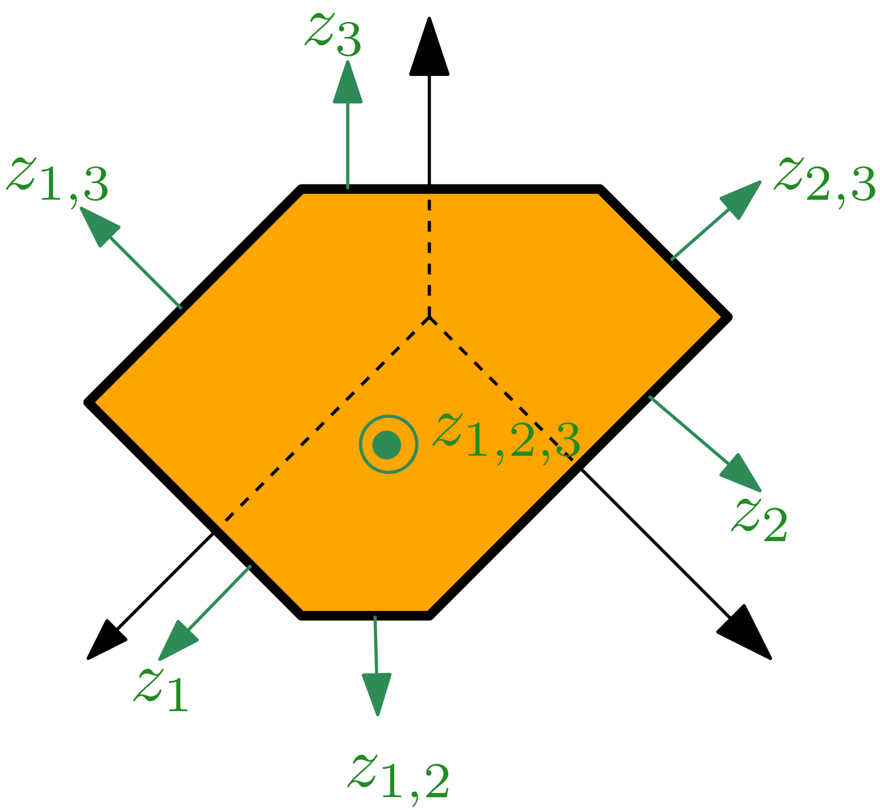
- **Duality**: $\mathbf{T}_{-P}(x, y) = \mathbf{T}_P(y, x)$.
- **Brylawski identities**:
  $\deg(\mathbf{T}_P(x, y)) = n)$ and $[x^k y^{n-k}]\mathbf{T}_P(x, y) = \binom{n}{k}$.

  Cor:[Brylawski 72] For any matroid $M \subseteq 2^{[n]}$, the coefficients $t_{i,j} = [x^i y^j]T_M(x, y)$ satisfy

$$\forall p < n, \quad \sum_{i=0}^{p} \sum_{j=0}^{i} \binom{p-i}{j} (-1)^j \ t_{i,j} = 0.$$

Universal Tutte Polynomial



Universal Tutte polynomial

Thm [BKP]. The Tutte polynomial is polynomial in the rank function.

Universal Tutte polynomial

Thm [BKP]. Let $n \in \mathbb{Z}_{>0}$, and let $\mathbf{z} = (z_S)_{\emptyset \neq S \subseteq [n]}$ be variables.
There exists a unique polynomial $\mathbf{T}_n(x, y; \mathbf{z})$ such that for all polymatroids on $[n]$,

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where f_P is the rank funtion of P.

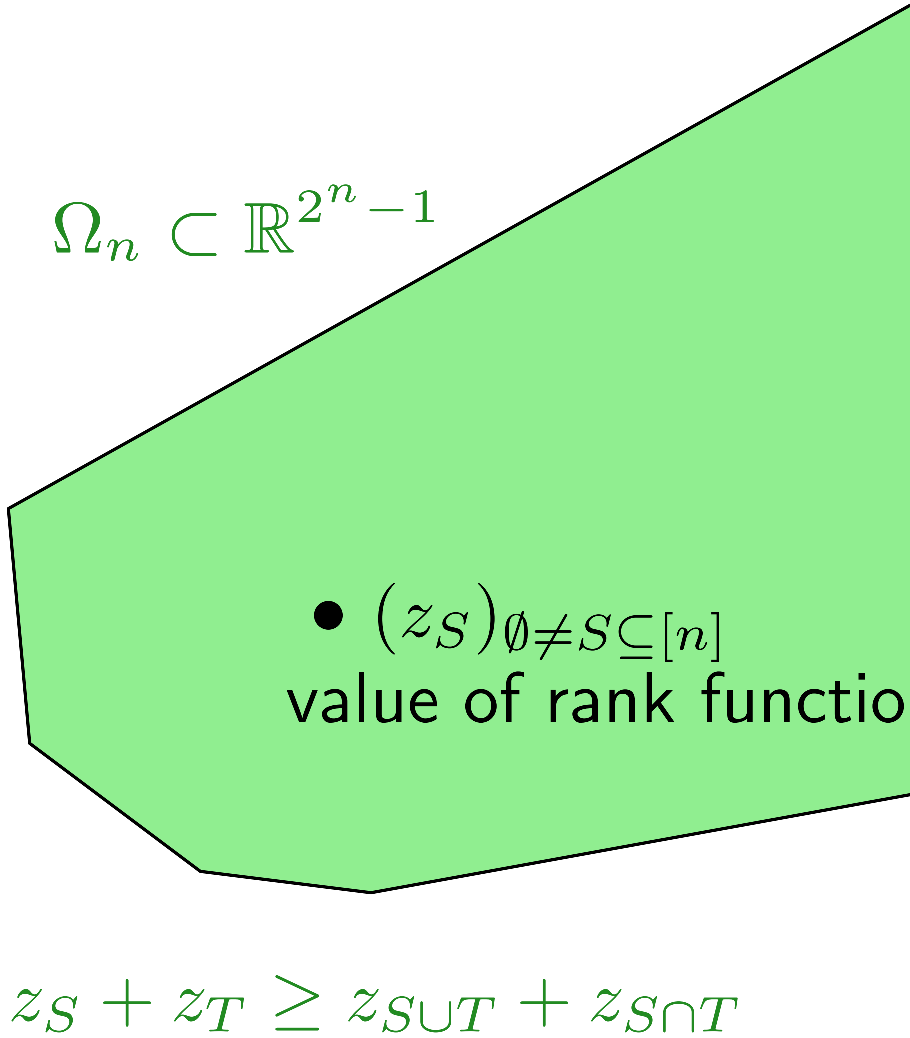
Example: n=3

$$\begin{aligned}\frac{\mathbf{T}_3(x, y; \mathbf{z})}{x + y - 1} =& x^2 + 2xy + y^2 \\ &+ (z_1 + z_2 + z_3 - z_{123} - 2)\, x \\ &+ (z_{12} + z_{13} + z_{23} - 2z_{123} - 2)\, y \\ &+ \tfrac{1}{2}(z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2 - z_1^2 - z_2^2 - z_3^2) \\ &- z_{123}(z_1 + z_2 + z_3) \\ &+ (z_1 z_{12} + z_1 z_{13} + z_2 z_{12} + z_2 z_{23} + z_3 z_{13} + z_3 z_{23}) \\ &+ \tfrac{1}{2}(3z_{123} - z_{12} - z_{13} - z_{23} - z_1 - z_2 - z_3) + 1.\end{aligned}$$

Proof:

Uniqueness:

Space Ω_n of polymatroids on $[n]$ contains a cone of dimension $2^n - 1$.



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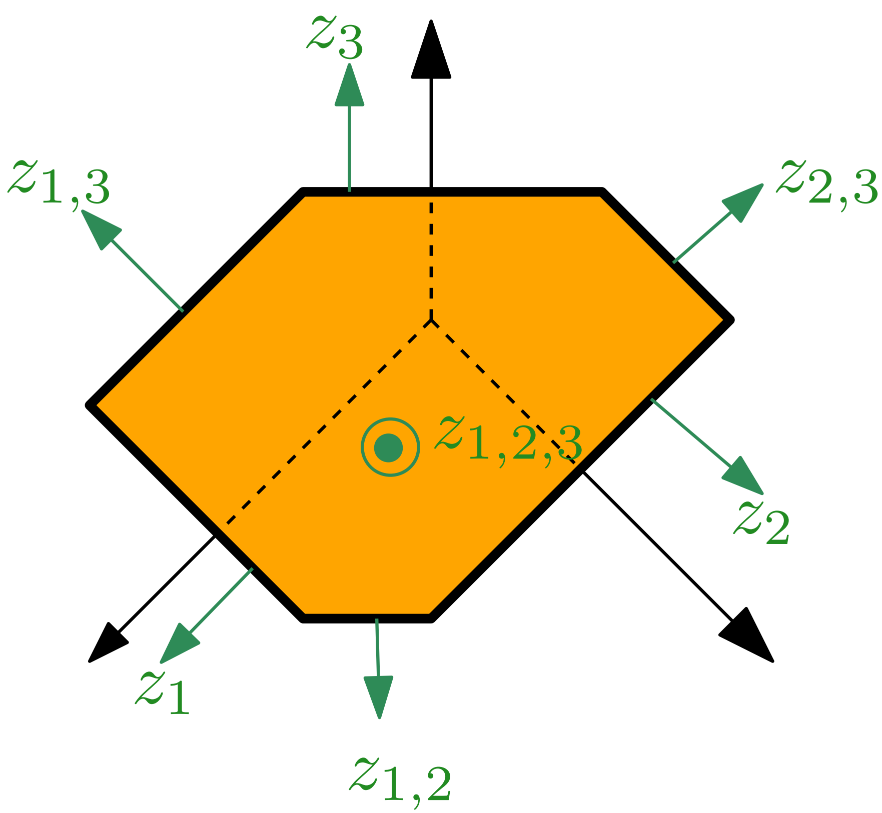
Existence:
• In the bulk of Ω_n: activity constant in the interior of each face, and number of points in each face is polynomial in $(z_S)_{S\subseteq[n]}$.



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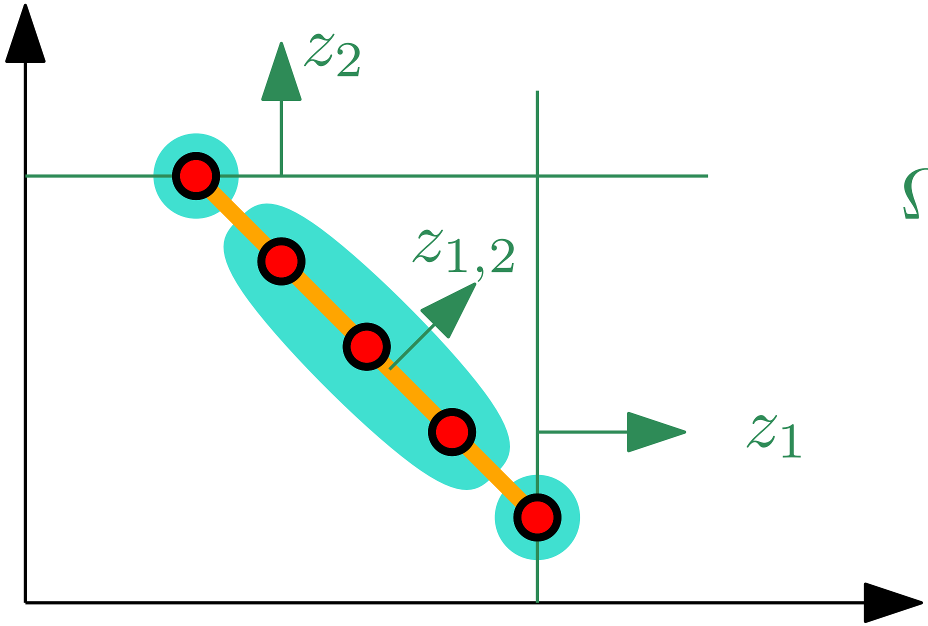
• At the boundary of Ω the contribution of "collapsing" faces behaves polynomially.

$n = 2$



$\Omega_2 : \; z_{1,2} \leq z_1 + z_2$

$$\mathbf{T}_2(x, y; \mathbf{z}) = (x + y - 1)x + (x + y - 1)y + (z_1 + z_2 - z_{1,2} - 1)(x + y - 1)$$

Explicit formula for $\mathbf{T}_n$

Def:[Postnikov] $(d_I)_{\emptyset \neq I \subseteq [n]} \in \mathbb{Z}_{\geq 0}^{2^n}$ is **draconian** if

$$\forall I_1, \ldots, I_k \subseteq [n], \quad d_{I_1} + \cdots + d_{I_k} \leq |I_1 \cup \cdots \cup I_k| - 1,$$

and

$$\sum_{I \subseteq [n]} d_I = n - 1.$$

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The **dragon polynomial** is the following polynomial in $\mathbf{t} = (t_I)_{\emptyset \neq I \subseteq [n]}$

$$D_n(\mathbf{t}) = \sum_{(d_I) \text{ draconian}} \binom{t_{[n]} - 1}{d_{[n]}} \prod_{\emptyset \neq I \subsetneq [n]} \binom{t_I}{d_I},$$

where $\binom{t}{d} := \frac{t(t-1)\cdots(t-d+1)}{d!}$.

Explicit formula for $\mathbf{T}_n$

Thm [BKP]:

The reparametrization $\widehat{\mathbf{T}}_n(x, y; \mathbf{t}) := \mathbf{T}_n(x, y; \mathbf{z})_{|z_I = \sum_{J \subseteq [n],\ J \cap I \neq \emptyset} t_J}$ has the following explicit formula:

$$\widehat{\mathbf{T}}_n(x, y; \mathbf{t}) = (x + y - 1) \sum_{\substack{B=(B_1,\ldots,B_\ell) \\ \biguplus B_k = [n]}} (-1)^{\ell-1} D_n(\mathbf{t}^B)\, x^{lr(B)-1}\, y^{rl(B)-1},$$

where

- $\mathbf{t}^B = (t_I^B)$ with $t_I^B = \left| \begin{array}{l} \sum_{J \subseteq \bigcup_{i<k} B_i} t_{I \cup J} \text{ if } I \subseteq B_k \text{ for some } k \\ 0 \text{ otherwise} \end{array} \right.$,
- $lr(B)$ is the number of left-to-right minima of B,
- $rl(B)$ is the number of right-to-left minima of B.

Some explanation/intuition for the formula:

$$\widehat{\mathbf{T}}_n(x,y;\mathbf{t}) = (x+y-1) \sum_{\substack{B=(B_1,\ldots,B_\ell) \\ \biguplus B_k=[n]}} (-1)^{\ell-1}\, D_n(\mathbf{t}^B)\, x^{lr(B)-1}\, y^{rl(B)-1},$$

- Change of variables $\mathbf{z} \to \mathbf{t}$:

The tuple $\mathbf{z} = (z_I)$ given by $z_I = \sum_{J\subseteq[n]\,:\,J\cap I\neq\emptyset} t_J$ is the rank function of $\mathcal{P} = \sum_{I\subseteq[n]} t_I\, \Delta_I$, where $\Delta_I = \mathrm{conv}(\mathbf{e}_i,\ i\in I)$.

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• The partitions B indexes the faces of a generic permutahedron. The tuple $\mathbf{t}^B$ gives the rank function of the face.

• The dragon polynomial $D_n(\mathbf{t})$ gives the number of lattice points in the interior of a permutahedron [Postnikov 06].

The draconian sequences correspond to the hypertrees of the complete hypergraph H_n on $[n]$ having one hyperedge for each $I \subseteq [n]$.

Application: Tutte polynomial of zonotopes

Example: The classical permutahedron

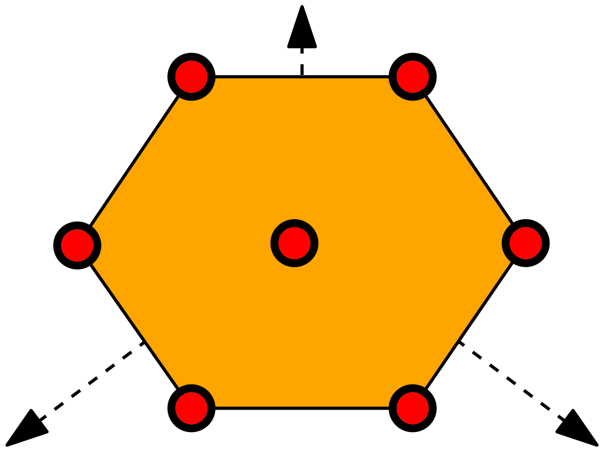
$$P_n = \text{conv}\{(\pi(1), \pi(2), \ldots, \pi(n)),\ \pi \in \mathfrak{S}_n\} \cap \mathbb{Z}^n$$

has Tutte polynomial

$$\mathbf{T}_{P_n}(x,y) = \sum_{F \text{ forest on } [n]} (x+y-1)^{\#\text{ connected components}}.$$

n=3:

P_3



$$\mathbf{T}_{P_3}(x,y) = (x+y-1)^3 + 3(x+y-1)^2 + 3(x+y-1).$$

Thanks.