Geometric equations for Matroid Varieties

Jt w/ W. Traves & Ashley Wheeler

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 \end{bmatrix} \\ \begin{matrix} 1 & \;2 & \;3 & \;4 & \;5 & \;6 & \;7 \end{matrix} \end{array}$$

$E = \{1, 2, \dots, 7\}$, $\mathcal{I} = \{ I \subseteq E \mid \{A_i \mid i \in I\} \text{ lin. ind.} \}$

$M_A = (E, \mathcal{I})$ is a <u>matroid</u>

<u>Q</u>: How can we describe $\{ B \text{ } 3\times 7 \text{ matrix} \mid M_B = M_A \}$?

- $\{1,2,7\} \notin \mathcal{I} \Rightarrow |A_1 A_2 A_7| = 0$ deg 3 poly in 9 entries
  $:= [127]$
- $[347], [567] = 0$
- $f = [134][256] - [234][156]$ must vanish on any matrix B s.t. $M_B = M_A$.
  mysterious! deg 6 poly in 18 entries.

Our setting:

- $X \subseteq \mathbb{C}^n$, $\dim X = k \longleftrightarrow$ pt in $G(n,k)$
- A_X = $k \times n$ matrix w/ rowspace X
- $M^X_X = M_{A_X}$

$$G(n,k) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$$

$\binom{n}{k}$-vector of $\underbrace{k \times k \text{ minors}}_{\text{brackets}}$ of A_X

= Plücker coords of X

$X \in G(n,k)$

$$\Gamma_X := \{ y \in G(n,k) \mid M_y = M_X \}$$

Gel'fand, Goresky, MacPherson, Serganova

$V_X := \overline{\Gamma_X}$ = matroid variety of X

$I_X := \{$polys in brackets vanishing on all of $V_X\}$

Q: Which polynomials are in I_X?

- $N_X := \langle [\text{non-bases}] \rangle \subseteq I_X$
- $N_X = I_X$ if M_X is a positroid
  Knutson, Lam, Speyer
- V_X can have any $\mathbb{Q}$-singularity

Mnëv, Sturmfels

positroids   matroid varieties $\longrightarrow$ abyss
w/ geometry & combinatorics

Grassmann-Cayley Algebra

- exterior algebra on $\mathbb{C}^3$ + extra operation
- join: exterior product written $x, y \in \mathbb{C}^3$

$$x \vee y = xy$$

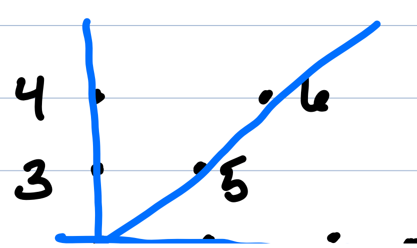
- meet: $\underbrace{\underset{\text{line}}{xy} \wedge \underset{\text{line}}{wz}}_{\text{pt}} = [xwz]y - [ywz]x$ pt $\in \mathbb{P}^2$

Ex: $(\underbrace{\underset{\text{line}}{12} \wedge \underset{\text{line}}{34}}_{\text{pt}}) \vee (\underset{\text{line}}{56}) = ([134]2 - [234]1) \vee 56$

$= [134][256] - [234][156]$

$= f$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 \end{bmatrix} \\ \begin{array}{ccccccc} 1 & & 3 & 4 & 5 & 6 & 7 \end{array} \end{array}$$



Find: $I_X = \langle [127], [347], [567], f \rangle$

$\supseteq$

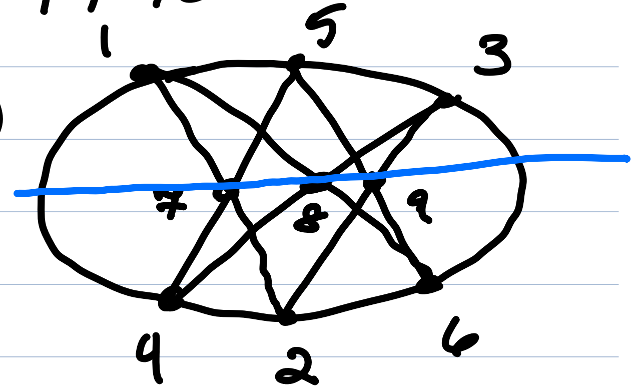
7 1 2 $\subseteq$?

Q: Are there any other interesting V_X w/ eqns derived via geometry?

<u>Pascal's theorem</u> $1,2,3,4,5,6 \in \mathbb{P}^2$ lie on a conic $\Leftrightarrow$

$(12 \wedge 45) \vee (16 \wedge 34) \vee (23 \wedge 56) = 0$

($\Leftarrow$ Braikenridge & Maclaurin)



A_X = 3×9 matrix whose cols are 9 pts from Pascal's thm

M_{A_X} has 7 non-bases: $[127], [457], \ldots \in N_X \subseteq I_X$

$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 16)$

$= [145][256][316][234] - [145][356][416][213]$

$- [245][256][316][134]$ deg 4 in brackets (deg 12 poly)

There are deg 3 & 2 polys as well!

- $7 \vee (23 \wedge 56) \vee (34 \wedge 16)$ deg 3 in brackets
- $7 \vee 8 \vee (34 \wedge 16)$ deg 2 in brackets

Generalizations

- more points on a conic in $\mathbb{P}^2$.
- points on higher degree curves in $\mathbb{P}^2$ (Cayley–Bacharach thm)
- Caminata–Schaffler '19 generalized Pascal's theorem to rational normal curves using the Grassmann–Cayley algebra.