# The h-vector of a matroid complex, paving matroids and the chip firing game

Criel Merino

Instituto de Matemáticas, UNAM

Matroids: Combinatorics, Algebra and Geometry October 13th 2022

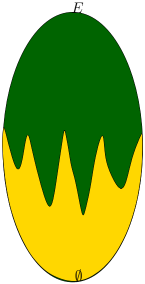
# Matroids: Definition

A *matroid* is an ordered pair $M = (E, \mathcal{I})$ such that $E$, the ground set of $M$, is a finite set and $\mathcal{I}$ is a collection of subsets of $E$, called *independent sets*, satisfying

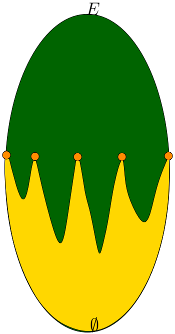
I1) $\emptyset \in \mathcal{I}$;

I2) if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;

I3) if $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, $\exists e \in I_2 - I_1$ s.t. $I_1 + e \in \mathcal{I}$.

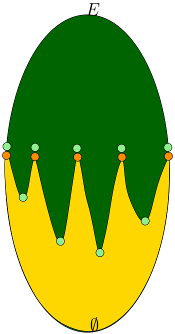


# Matroids: Bases and circuits




Maximal independents sets are called *bases* and by I3 all bases have the same cardinality. The common cardinality is called the rank of the matroid and its denoted $r(M)$.

# Matroids: Bases and circuits




Minimal subsets of $E$ that are not independents are called *circuits*.

# Simplicial complex

Associated to any $(d-1)$-dimensional simplicial complex $\Delta$ we have its *face enumerator*

$$f_\Delta(x) = \sum_{F \in \Delta} x^{d-|F|} = \sum_{i=0}^{d} f_i x^{d-i},$$

and the corresponding *f-vector* $(f_0, f_1, \ldots, f_d)$. A simplicial complex is *pure* if all its facets have the same cardinality.

# Simplicial complex

Associated to any $(d-1)$-dimensional simplicial complex $\Delta$ we have its *face enumerator*

$$f_\Delta(x) = \sum_{F\in\Delta} x^{d-|F|} = \sum_{i=0}^{d} f_i x^{d-i},$$

and the corresponding *f-vector* $(f_0, f_1, \ldots, f_d)$. A simplicial complex is *pure* if all its facets have the same cardinality.

**Theorem (Lenz 2013)**

*The sequence $(f_0, f_1, \ldots, f_d)$ is log-concave for the matroid complex of a representable matroid.*

# Simplicial complex

For a pure simplicial complex $\Delta$, a *shelling* is a linear order of the facets $F_1, F_2, \ldots, F_t$ such that each facets meets the complex generated by its predecessors in a non-void union of maximal proper faces. A complex is said to be *shellable* if it is pure and admits a shelling.
Let $\Delta_i$ the subcomplex generated by the facets $F_1, \ldots, F_i$ and let $\mathcal{R}(F_i)$ be the *unique* minimal face of $F_i$ which lies in $\Delta_i - \Delta_{i-1}$.

# Simplicial complex

Associated to any (d-1)-dimensional shellable simplicial complex $\Delta$ we have its *shelling polynomial*

$$h_\Delta(x) = \sum_{i=1}^{t} x^{d-|\mathcal{R}(F_i)|} = \sum_{i=0}^{d} h_i x^{d-i},$$

and the corresponding *h-vector* $(h_0, h_1, \ldots, h_d)$.

# Simplicial complex

Associated to any (d-1)-dimensional shellable simplicial complex $\Delta$ we have its *shelling polynomial*

$$h_\Delta(x) = \sum_{i=1}^{t} x^{d-|\mathcal{R}(F_i)|} = \sum_{i=0}^{d} h_i x^{d-i},$$

and the corresponding *h-vector* $(h_0, h_1, \ldots, h_d)$.

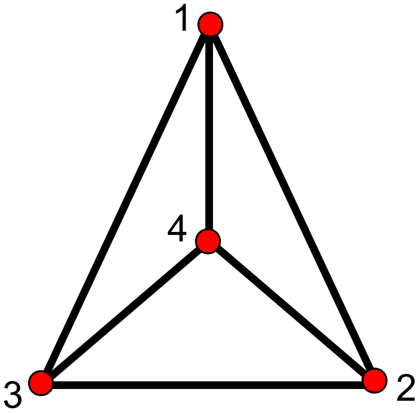
**Theorem (Huh 2015)**

*The sequence $(h_0, h_1, \ldots, h_d)$ is log-concave for the matroid complex of a matroid representable over a field of characteristic zero.*

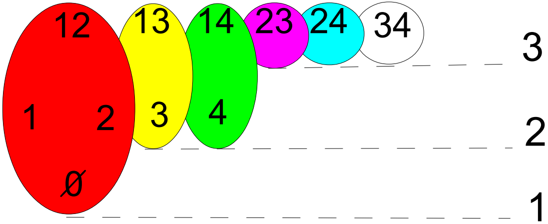
# Example

(1,4,6)

$f_\Delta = x^2 + 4x + 6$



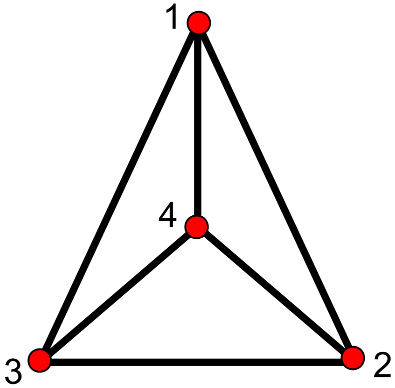
# Example



# Example

(1,2,3)

$h_\Delta = x^2 + 2x + 3$



# Simplicial complex

**Theorem (McMullen;1970)**

$$f_\Delta(x) = h_\Delta(x+1).$$

Thus, we obtain that

**Observation**

$$h_0 + h_1 + \cdots + h_d = f_d.$$

$$h_k = \sum_{i=0}^{k} (-1)^{i+k} \binom{d-i}{k-i} f_i.$$

# Matroid complexes

Let $M = (E, \mathcal{I})$ be a matroid, the family $\mathcal{I}$ forms a simplicial complex $\Delta(M)$ of dimension $r(M) - 1$, called *matroid complex*. The facets of $\Delta(M)$ are the bases of $M$ and therefore $\Delta(M)$ is pure.

**Theorem (Provan 1977)**

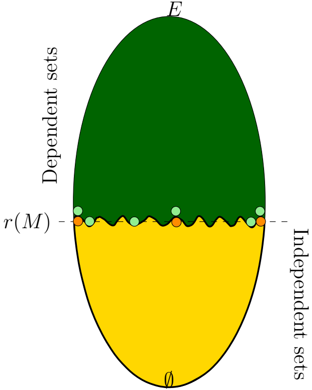
*The matroid complex $\Delta(M)$ is shellable.*

# Paving matroid

A matroid $M$ is *paving* if all it's circuits have size at least $r(M)$

# Paving matroid

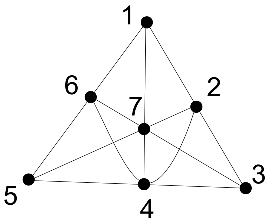
A matroid $M$ is *paving* if all it's circuits have size at least $r(M)$

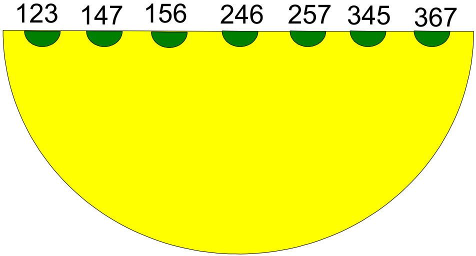


# Paving matroid

A matroid $M$ is *paving* if all it's circuits have size at least $r(M)$

Fano



# Importance of paving matroids

1973 J. E. Blackburn, H. H. Crapo, and D. A. Higgs. A catalogue of combinatorial geometries.

1976 Dominic Welsh ask if most matroids are paving.

2008 D. Mayhew and G.F. Royle. Matroids with nine elements

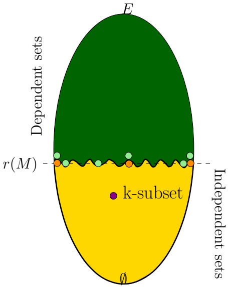
2010 D. Mayhew, M. Newman, D. Welsh, and G. Whittle. Conjecture that asymptotically most matroids are paving. $\lim_{n\to\infty} \frac{s_n}{m_n} = 1$

2015 R. A. Pendavingh, J. G. Van Der Pol. $\log m_n \leq (1 + o(1)) \log s_n$ as $n \to \infty$

# h-vector for paving matroids

How is the h-vector of a paving matroid?.

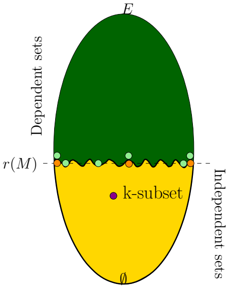
$$f\text{-vector is } \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{r-1}, b(M)\right)$$



# h-vector for paving matroids

How is the h-vector of a paving matroid?.

$$f\text{-vector is } \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{r-1}, b(M)\right)$$




Note: It is enough to consider paving matroids with no loops nor coloops.

# h-vector for paving matroids

Using the relation between the $f_i$'s and $h_i$'s, the $h$-vector of a paving matroid is

# h-vector for paving matroids

Using the relation between the $f_i$'s and $h_i$'s, the $h$-vector of a paving matroid is

$$(\binom{n-r-1}{0}, \binom{n-r}{1}, \ldots, \binom{n-2}{r-1}, b(M) - \binom{n-1}{r-1}).$$

It only rest to know

$$h_r = b(M) - \binom{n-1}{r-1}.$$

# Multicomplex

Note that

$$h_k = \binom{n - r - 1 + k}{k}.$$

This expression correspond to the number of monomials over $n - r$ variables with degree $k$, for $0 \leq k \leq r - 1$.

# Multicomplex

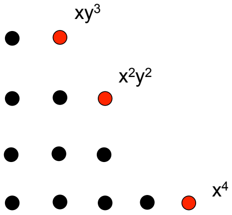
Let us consider the ring $\mathbb{Z}[z_1, \ldots, z_d]$. The set of all (monic) monomials over $z_1, \ldots, z_d$ is a poset with the divisibility relation.

# Multicomplex

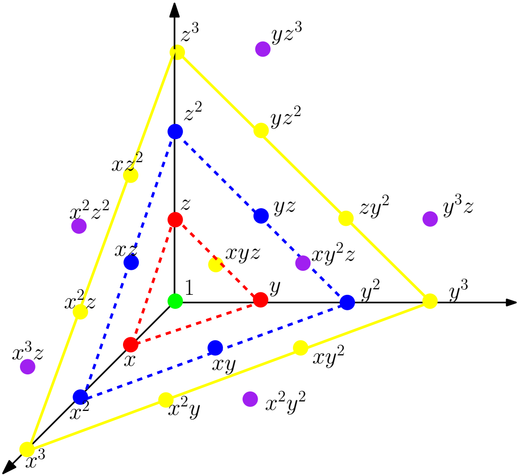
Let us consider the ring $\mathbb{Z}[z_1, \ldots, z_d]$. The set of all (monic) monomials over $z_1, \ldots, z_d$ is a poset with the divisibility relation.

## Definition

A *multicomplex* $\Sigma$ is a subset of monomials (in this poset ) which is closed under divisibility. If all the maximal elements of $\Sigma$ have the same degree we said that $\Sigma$ is *pure*.



# Example: 3 variables, maximal degree 4



# Multicomplexes

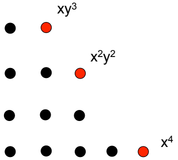
**Definition**

An integer sequence $(h_0, h_1, \ldots)$ is an $O$-sequence if there exists a multicomplex with $h_i$ monomials of degree $i$. The $O$-sequence is pure if the multicomplex is pure.

# Multicomplexes

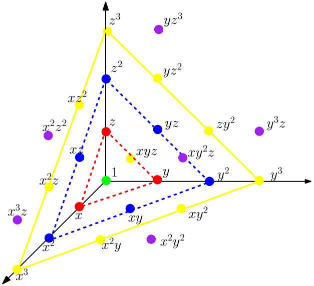
**Definition**

An integer sequence $(h_0, h_1, \ldots)$ is an $O$-sequence if there exists a multicomplex with $h_i$ monomials of degree $i$. The $O$-sequence is pure if the multicomplex is pure.




the pure $O$-sequence here is (1,2,3,4,3).

# Example




Here the pure O-sequence is (1,3,6,10,6).

# Stanley's conjecture on h-vectors

**Conjecture (R. Stanley 1977)**

*The h-vector of a matroid complex is a pure O-sequence.*

# Stanley's conjecture on h-vectors

- Cographic matroids. Biggs-Merino1997.
- Lattice-path matroids. Schweig 2010
- Graphic matroids of coned graphs. Kook 2012;
- Paving matroids Merino, Noble, Ramírez-Ibañez and Villarroel-Flores, 2012
- Rank 3 matroids, corank 2 matroids, and all matroids with at most 9 elements. De Loera, Kemper, and Klee 2012

# Stanley's conjecture on h-vectors

**Conjecture**

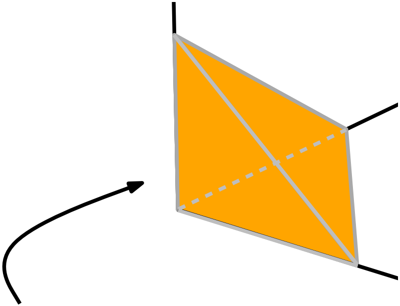
*The h-vector of a matroid complex is a pure O-sequence.*

- Rank 3 matroids, Há, Stokes, and Zanello 2013
- Cotransversal matroids Oh 2013; Sarmiento 2018
- Truncation of a matroid, generalized Catalan matroids, rank $d$ matroids with $h_d \leq 5$, Constantinescu, Kahle, and Varbaro 2014
- rank 4 matroids, Kleen and Samper 2015
- Graphic matroids of biconed graphs. Cranford, Dochterman, Haithcock , Marsh,. Oh, and Truman 2021.

# Stanley's conjeture and paving matroids

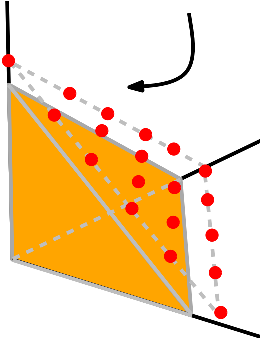
To prove the conjecture for paving matroids it is enough to prove that there is a pure multicomplex for which the quantity $h_r$ fits to get a pure multicomplex from the one that we already have construct for $h_0, h_1, \ldots, h_{r-1}$.

# Proof idea



We define the multicomplex $\mathcal{M}_{r,d}$ to be the pure multicomplex in which the maximal elements are all the monomials of degree $r$ in $d$ indeterminates $z_1, \ldots, z_d$. This multicomplex gives an O-sequence $(h_0, \ldots, h_r)$, where $h_k = \binom{d+k-1}{k}$.

# Proof idea

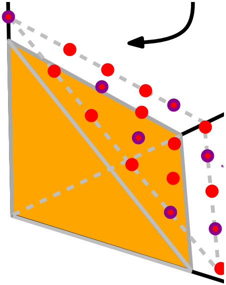


Now, let us define

$$g(r,n) = \min\left\{ b(M) - \binom{n-1}{r-1} \mid M \in \mathcal{P}_{r,n} \right\}.$$

Observe that $g(r,n)$ equals the minimum value of $h_r$ among all $h$-vectors of matroids in $\mathcal{P}_{r,n}$.

# Proof idea



Finally, take

$$f(r,d) = \min\{h_r \mid (h_0, \dots h_r) \text{ is the pure O-sequence of } \mathcal{M} \supset \mathcal{M}_{r-1,d}\}.$$

Thus, to prove Stanley's conjecture for paving matroids is enough to show that $g(r,n) \geq f(r,n-r)$.

# Proof idea

**Theorem (Merino,Noble,Ramírez, Villarroel)**

*For all rank-$r$ loopless and coloopless paving matroid $M$ with $n$ elements we have that $g(r, n)$ is at least $f(r, n - r)$.*

**Proof.**

- paving matroids are closed under minors;
- Contraction-deletion
- Careful induction on $r + n$;
- structural theorem for rank-$r$ coloopless paving matroid $M$ such that every element $e$ of $M$ , $M \setminus e$ has a coloop.

□

# Extremal multicomplex

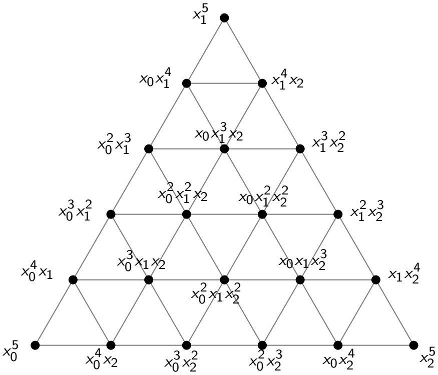
Let $X_{r,d}$ be the set of (monic) monomials over $d$ variables of degree $r$. The cardinality of $X_{r,d}$ is $\binom{d+r-1}{r}$.

**Definition**

Let $G_{r,d}$ be the the graph with vertices the elements in $X_{r,d}$ and two monomials $m$ and $m'$ are adjacent if there exists $i \neq j$ such that $z_i m = z_j m'$.

# Example

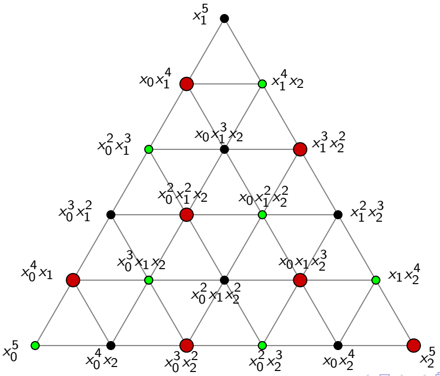
Here is $G_{5,3}$ with $\binom{7}{5}$ vertices.



# Standard colouring

We define the *standard colouring* of $G_{r,d}$ as the colouring that assigns to $z_0^{a_0} \cdots z_{d-1}^{a_{d-1}}$ the colour $(0, \ldots, d-1) \cdot (a_0, \ldots, a_{d-1})$ mod $d$. It is easy to check that this is a proper colouring. We also define

$$\overline{f}(r, d) = \text{minimum size of a chromatic class.}$$



# Extremal multicomplexes $f(r,d)$ and $\overline{f}(r,d)$

**Conjetura MNRV**

$f(r,d) = \overline{f}(r,d) = L_2(r,d)$.

Where $L_2(r,d)$ is the number of aperiodic binary necklaces with $r$ white and $d$ black beads. Example

# The second part of the conjecture: Number theory

In "Combinatorics of necklaces and "hermite reciprocity" ", Elashvili, Jibladze and Pataraiase study the quantity $a_k(r,d)$, the number of no negative solutions to the system:

$$\sum_{j=0}^{d-1} j\lambda_j \equiv k \pmod{d}; \ \sum_{i=0}^{d-1} \lambda_i = r;$$

The number of monomials $G_{r,d}$ with color $k$ is $a_k(r,d)$. They results prove (indirectly). Thanks to Tristram Bogart for pointing this out.

**Corollary**

$L_2(r,d) = a_1(r,d) = \overline{f}(r,d).$

# First part of conjecture: partial results

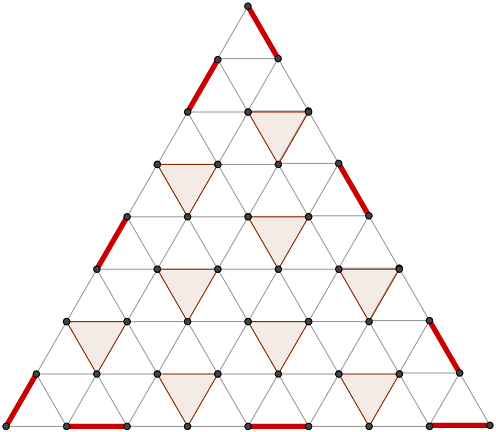
We have the following results.

**Theorem**

$f(r,d) = \overline{f}(r,d)$ *for the values*

1. $d = 2, 3, 4$ *all* $r \geq 1$
2. $r = 2, 3, 4$ *all* $d \geq 1$
3. $r = 5$, $d$ *odd.*

# Second part of conjecture: partial results



# Counting bases in graphic matroids

Let $M$ be the graphic matroid of a connected graph $G$ with $n+1$ vertices. The *Laplacian* of $G$ is the $(n+1)\times(n+1)$ matrix $L(G)=L$ given by

$$L_{i,j} = \begin{cases} -a(v_i,v_j), & \text{if } i \neq j, \\ deg(v_i), & \text{if } i = j, \end{cases}$$

where $a(v_i,v_j)$ denotes the number of edges joining $v_i$ and $v_j$, and $deg(v_i)$ denotes the number of non-loop edges incident to $v$. The *reduced Laplacian* of $G$, $L^q(G)$, is obtained from $L(G)$ by deleting the row and column corresponding to a vertex $q$ in $G$.

**Theorem (Kirchhoff 1847; Sylvester 1857; Borchardt 1860; Maxwell 1892)**

*The number of bases of $M(G)$ is given by*

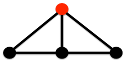
$$b(M) = det(L^q(G))$$

# The chip firing game

We consider the equivalence relation over $\mathbb{Z}^n$, where two configurations $\sigma_1$ and $\sigma_2$ are related iff $\sigma_1 - \sigma_2 \in < L^q(G) >$ The number of equivalent clases is $det(L^q(G))$. But which representative in each class can give us an interesting combinatorial object?

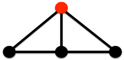
# The chip firing game

The rows of $L^q$ are $R_{v_1}, \ldots, R_{v_n}$. Each of these rows define a rule: You can add $-R_{v_i}$ to a configuration $\sigma$ if $\sigma - R_{v_i} \geq 0$. We also have a special rule corresponding to vertex $q$ that consist of adding vector equal to the sum of the rows of $L^q$. If no normal rule applies, we use the special rule.



| | | |
|---|---|---|
| 2 | -1 | 0 |
| -1 | 3 | -1 |
| 0 | -1 | 2 |
| 1 | 1 | 1 |

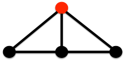
# The chip firing game



-2 1 0

| | | |
|---|---|---|
| 3 | 5 | 1 |

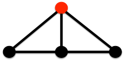
# The chip firing game



1 -3 1

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |

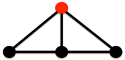
# The chip firing game



-2 1 0

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |

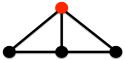
# The chip firing game



0 1 -2

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |

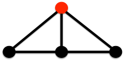
# The chip firing game



1 -3 1

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |

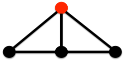
# The chip firing game



1 1 1

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |

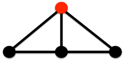
# The chip firing game



-2 1 0

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |
| 2 | 3 | 2 |

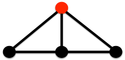
# The chip firing game



0 1 -2

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |

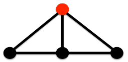
# The chip firing game



1 -3 1

| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |

# The chip firing game



| | | |
|---|---|---|
| 3 | 5 | 1 |
| 1 | 6 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |
| 2 | 3 | 2 |
| 0 | 4 | 2 |
| 0 | 5 | 0 |
| 1 | 2 | 1 |

# The chip firing game

Critical configurations are configurations that are stable, i.e. entry $v$ is at most $deg(v) - 1$, and that they recur under the chip firing game.

- There is unique critical configuration in each congruence class

# The chip firing game

Critical configurations are configurations that are stable, i.e. entry $v$ is at most $deg(v) - 1$, and that they recur under the chip firing game.

- There is unique critical configuration in each congruence class
- There is a maximal critical configuration $\sigma_m$, where $\sigma_m(v) = deg(v) - 1$ for all $v \neq q$.

# The chip firing game

Critical configurations are configurations that are stable, i.e. entry $v$ is at most $deg(v) - 1$, and that they recur under the chip firing game.

- There is unique critical configuration in each congruence class
- There is a maximal critical configuration $\sigma_m$, where $\sigma_m(v) = deg(v) - 1$ for all $v \neq q$.
- The chip firing game (sandpile model) have appeared independently in Mathematics and Physics.

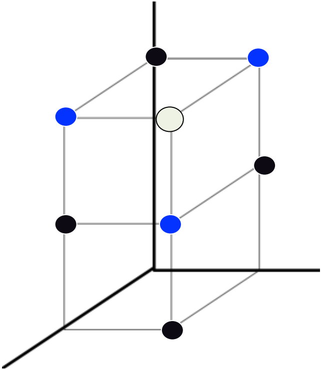
# The chip firing game

Associated to each critical configuration is the weight of a configuration $\sigma$, $w(\sigma)$, define as the sum of its entries. If $\sigma$ is a critical configuration, we define its level as $\mathrm{level}(\sigma) = w(\sigma) - |E(G)| + deg(q)$. Then, $0 \leq \mathrm{level}(\sigma) \leq |E(G)| - |V(G)| + 1 = r^*(M(G))$.
Let $G = (V, E)$ be a graph with $q \in V$ and, for $i \geq 0$, let $c_i(q)$ be the number of critical configurations with level $i$ in the chip-firing game. Then the critical configuation polynomial (generating function of critical configurations by level) is

$$P(G, q; y) = \sum_{i=0}^{r^*(M(G))} c_i(q) y^i.$$

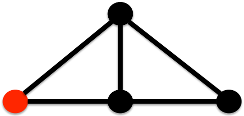
# The chip firing game

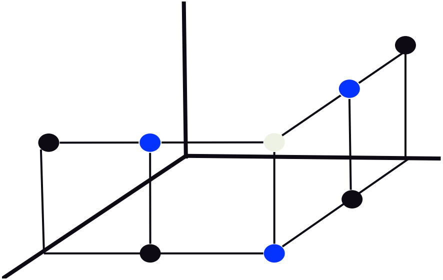
Example $K_4 \setminus e$: the critical configurations (1,2,1) has level 2; configurations (0,2,1), (1,1,1) and (1,2,0) have level 1; and (0,2,0), (0,1,1), (1,0,1) and (1,1,0) have level 0. Thus, the critical configuration polynomial is $y^2 + 3y + 4$.



# The chip firing game

Example $K_4 \setminus e$, different vertex $q$: Critical configurations (2,1,2) has level 2; (2,0,2), (2,1,1) and (1,1,2) have level 1; and (2,1,0), (0,1,2), (2,0,1) and (1,0,2) have level 0. The critical configuration polynomial is again $y^2 + 3y + 4$.





# The chip firing game

**Theorem (Merino 1997)**

*For a connected graph $G$ with special vertex $q$ we have that*

$$\sum_{i=0}^{r^*} h_i x^{r^*-i} = h_{\Delta(M^*(G))}(y) = P(G, q; y) = \sum_{i=0}^{r^*(M(G))} c_i(q) y^i.$$

# Proof idea

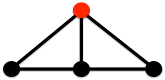
The shelling polynomial satisfies contraction-deletion:

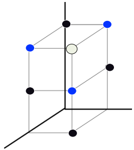
- $h_{\Delta(M)} = h_{\Delta(M\setminus e)} + h_{\Delta(M/e)}$ for $e$ not a loop of a coloop;
- If $e$ is a loop $h_{\Delta(M)} = h_{\Delta(M\setminus e)}$ ;
- if $e$ is a coloop $h_{\Delta(M)} = xh_{\Delta(M/e)}$;
- if $M$ is $U_{0,1}$, $h_{\Delta(M)}(x)$ is 1 and if $M$ is $U_{1,1}$, $h_{\Delta(M)}(x)$ is x.

# Proof idea

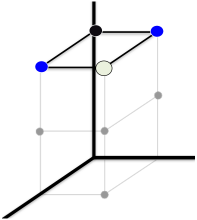
Fix an edge $\{q, v\}$, not a loop or coloop, with $v$ of degree $k$. The critical configurations of $G/e$ are (the projection of) the critical configurations with $v$-entry equal to $k-1$. While the critical configurations of $G \setminus e$ are the critical configurations with $v$-entry strictly less than $k-1$.

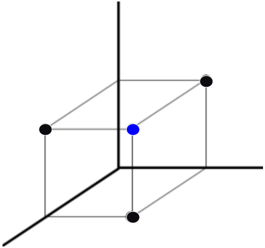
# Proof idea





Critical configurations with second entry equal to 2 are: (1,2,1), (0,2,1), (1,2,0) and (0,2,0) Critical configurations with second entry less than 2 are: (1,1,1), (0,1,1), (1,0,1) and (1,1,0)





# Proof idea

The set $\{\sigma_m - \sigma\}$ is a pure multicomplex.

- If $\sigma$ is a critical configuration and $\sigma'$ is stable with $\sigma \leq \sigma'$, then, $\sigma'$ is also critical.
- If $\sigma$ is a critical configuration, by marking the essential chips for game to recur, you can find a critical configuration $\sigma_0$ with $|E| - \deg(q)$ chips. This $\sigma_0$ has minimal level and $\sigma_0 \leq \sigma$

# A group from graphs

The set of critical configurations have a natural binary operation. If $\sigma_1$ and $\sigma_2$ are critical configurations, then we define $\sigma_1 \odot \sigma_2$ to be the unique critical configuration associated with the configuration $\sigma_1 + \sigma_2$. Critical configurations with this operation form an Abelian group called *critical group* or *sandpile group*. This group is isomorphic to the group $K(G) \cong \mathbb{Z}^n / \mathbb{Z}^n L^q(G)$, where $\mathbb{Z}^n L^q(G)$ is the integer row-span of the reduced Laplacian of $G$. [1]

[1] Berman 1986; Lorenzini 1989; Dhar 1990; Biggs 1996; Bacher, de la Harpe and Nagnibeda 1997.

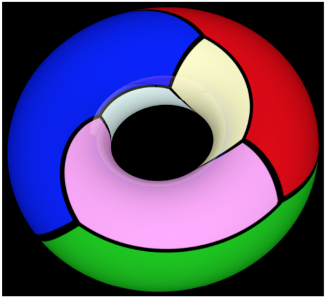
# A group from graphs

For example, the critical group of the complete graph $K_n$ is isomorphic to the direct sum of $n-2$ copies of $\mathbb{Z}/n\mathbb{Z}$. This generalizes the famous theorem by Cayley[2], $\tau(K_n) = n^{n-2}$.

[2] Cayley 1889; Sylvester 1857; Borchardt 1860

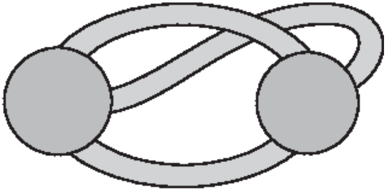
# A group for ribbon graphs

Given a 2-cellular embedding of a graph $G$ in a closed compact surface $\Sigma$, the ribbon graph $\mathbb{G}$ is obtained by taking a small neighborhood of the embedding of $G$ and deleting its complement.
Notice that we can always consider a ribbon graph $\mathbb{G}$ as an abstract graph $G$, by disregarding the information about the embedding.



# A group for ribbon graphs

A ribbon graph is a *quasi-tree* if it has only one face.
A ribbon graph is *orientable* if it is embedded in an orientable surface.



# A group for ribbon graphs

A $\Delta$-*matroid* $D = (E, \mathcal{F})$ consists of a finite set $E$ and a non-empty set $\mathcal{F}$ of subsets of $E$, called the *feasible sets*, that satisfies the following:

1. $\mathcal{F} \neq \emptyset$
2. For all $X, Y \in \mathcal{F}$, if there is an element $u \in X \triangle Y$, then there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$. *Symmetric Exchange Axiom*

Note that it may be the case that $u = v$.
Example: $B_3$ is the $\Delta$-matroid with groundset $E = \{a, b, c\}$ and the feasible sets are $\{a\}, \{b\}, \{c\}$ and $\{a, b, c\}$

$\Delta$-matroids were introduced by A. Bouchet in 1987.

# A group for ribbon graphs

The bases of a matroid are the feasible sets of a $\Delta$-matroid.
If you have a $\Delta$-matroid $D = (E, \mathcal{F})$ and $A \subseteq E$, the family of subsets

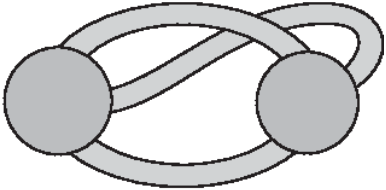
$$\mathcal{F} \triangle A = \{I \triangle A : I \in \mathcal{F}\}$$

is the set of feasible sets of the $\Delta$-matroid $D \triangle A$, the twist of $D$ by $A$.
$\Delta$-matroids are consider the same up twists.

# A group for ribbon graphs

**Theorem**

*The set of spanning quasi-trees of a connected ribbon graph is the set of feasible sets of a delta-matroid.*



Example: The $\Delta$-matroid is $B_3$ with groundset $E = \{a, b, c\}$ and the feasible sets are $\{a\}, \{b\}, \{c\}$ and $\{a, b, c\}$

# A group for ribbon graphs

Let $\mathbb{G} = (V, E)$ be a ribbon graph and $A \subseteq E$ a subset of edges. The *partial dual* $\mathbb{G}^A$ of $\mathbb{G}$ is obtained in the following way. Consider the spanning ribbon subgraph $\mathbb{H} = (V, A)$. Now, take $\mathbb{G}$ and glue a disk onto each boundary component of $\mathbb{H}$; these disks are the vertices of $\mathbb{G}^A$. Removing the interior of all old vertices of $\mathbb{G}$ we get $\mathbb{G}^A$.

The dual of $\mathbb{G}^*$ is the ribbon graph $\mathbb{G}^E$.

# A group for ribbon graphs

**Theorem (Merino, Moffatt, Noble, 2022)**

*For an orientable ribbon graph $\mathbb{G}$, there exists an abelian group $K(\mathbb{G})$ such that*

1. *The order of the group equals the number of spanning quasi-trees of $\mathbb{G}$;*

# A group for ribbon graphs

**Theorem (Merino, Moffatt, Noble, 2022)**

*For an orientable ribbon graph $\mathbb{G}$, there exists an abelian group $K(\mathbb{G})$ such that*

1. *The order of the group equals the number of spanning quasi-trees of $\mathbb{G}$;*
2. *If $\mathbb{G}$ is plana, $K(\mathbb{G}) \cong K(G)$;*

# A group for ribbon graphs

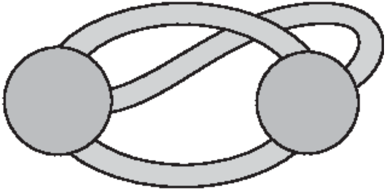
**Theorem (Merino, Moffatt, Noble, 2022)**

*For an orientable ribbon graph $\mathbb{G}$, there exists an abelian group $K(\mathbb{G})$ such that*

1. *The order of the group equals the number of spanning quasi-trees of $\mathbb{G}$;*
2. *If $\mathbb{G}$ is plana, $K(\mathbb{G}) \cong K(G)$;*
3. $K(\mathbb{G}^*) \cong K(\mathbb{G})$. *(in fact $K(\mathbb{G}^A) \cong K(\mathbb{G})$ )*

# A group for ribbon graphs

Example: The group of $B_3$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.



Thanks

**Lemma**

*Let $M$ be a rank-$r$ coloopless paving matroid. If for every element $e$ of $M$, $M \setminus e$ has a coloop, then one of the following three cases happens.*

1. *$M$ is isomorphic to $U_{r,r+1}$, $r \geq 1$.*
2. *$M$ is the 2-stretching of a uniform matroid $U_{s,s+2}$, for some $s \geq 1$.*
3. *$M$ is isomorphic to $U_{1,2} \oplus U_{1,2}$.*