Type polytopes and products of simplices

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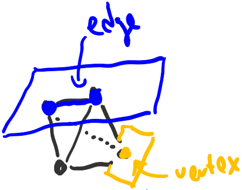
☆ Minkowski summands of cubes ☆

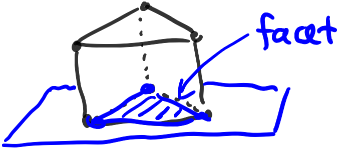
Polytopes

$P \subseteq \mathbb{R}^d$

- Convex hull of finitely many points
- Bounded intersection of finitely many halfspaces



3-simplex



triangular prism

Face: intersection of P with a supporting hyperplane

k-simplex: $k+1$ affinely independent points $\to$ take convex hull

Combinatorial isomorphism

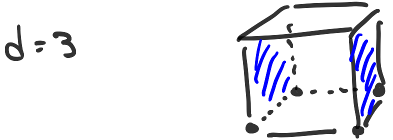
Face poset: Set of faces of P ordered by inclusion

P & Q are comb. iso. if face posets are isomorphic

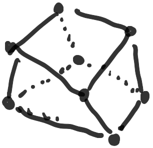
BIG QUESTION: If P and Q are combinatorially isomorphic, how different can they be?

Example: Cubes

Standard cube: $C_d = \left\{x \in \mathbb{R}^d \mid 0 \leq x_i \leq 1 \text{ for all } i \in [d]\right\}$

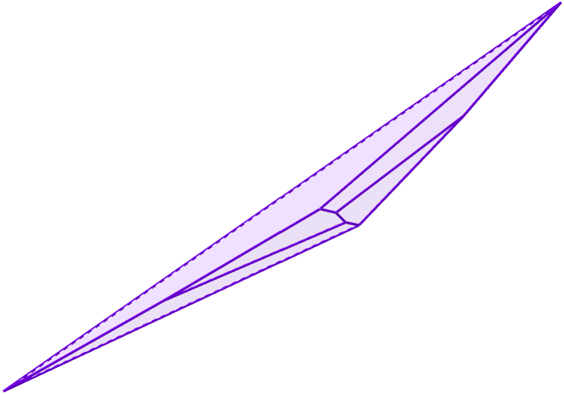


Klee–Minty cube: Perturb vertices—simplex algorithm might have to visit all 2^d vertices



Example: Cubes

For $d \geq 3$, there exist d-cubes for which each pair of opposing facets is **perpendicular**.



A more precise question

Realization space of P: Set of all polytopes that are combinatorially isomorphic to P

- Wild topology in general! ($d \geq 4$ any homotopy type could appr)
- Adiprasito, Kalmanovich, Nevo '19
  ↳ Realizatn space of d-cube is contractible

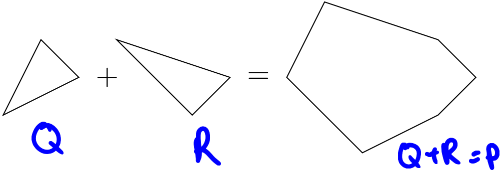
Type cone of P: Set of all polytopes that are combinatorially isomorphic to P with the same facet normal vectors

- Take closure ⇝ degeneracies on boundary
- Padrol, Palu, Pilaud, Plamadon '19
  ↳ Considering simplicial type cones

Minkowski sums and summands

Q, R polytopes in $\mathbb{R}^d$

$$Q+R = \{q+r \mid q \in Q,\ r \in R\}$$ ← Minkowski sum



Q is a (weak) Minkowski summand of P

if $\exists R$ (and scalar λ) s.t. $Q+R=(\lambda)P$

Minkowski summands...in other words

Theorem (Shephard)

Let $P = \{x \in \mathbb{R}^d : Ux \leq z\}$ be an irredundant inequality description for a polytope. The following are equivalent.

(i) *Q is a weak Minkowski summand of P.*

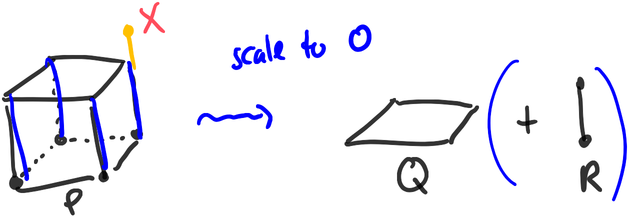
(ii) ***(Edge lengths)*** *There exists a map $\varphi : V(P) \to V(Q)$ such that for $v_i, v_j \in V(P)$ with $\{v_i, v_j\} \in E(P)$ we have $\varphi(v_i) - \varphi(v_j) = \lambda_{i,j}(v_i - v_j)$, for some $\lambda_{i,j} \in \mathbb{R}_{\geq 0}$.*

(iii) ***(Facet heights)*** *There exists an $\eta \in \mathbb{R}^m$ such that $Q = \{x \in \mathbb{R}^d : Ux \leq \eta\}$ and for any subset of rows S such that the linear system $\{\langle u_i, x\rangle = z_i, \forall i \in S\}$ defines a vertex of P, the linear system $\{\langle u_i, x\rangle = \eta_i, \forall i \in S\}$ defines a vertex in Q.*

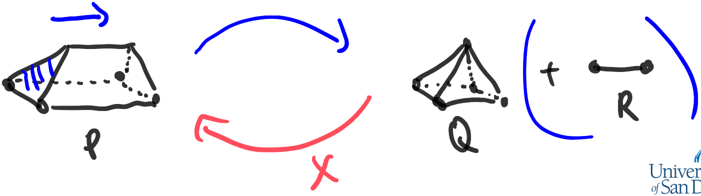
Those other words. . . in other words

Q is a Minkowski summand of P

Edge lengths



Facet heights



Type cones and type polytopes

Type cone of P: $\mathbb{TC}(P)$

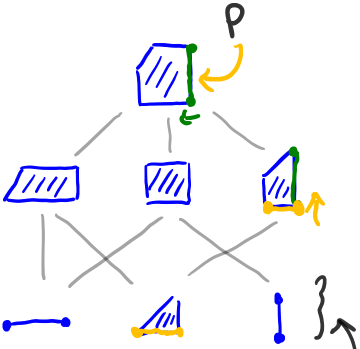
- set of all weak Minkowski summands of P up to translation
- each point $\mathbb{TC}(P)$ is a realization of P

Type polytope of P: $\mathbb{TP}(P)$

- Slice $\mathbb{TC}(P)$ w/ "appropriate" hyperplane

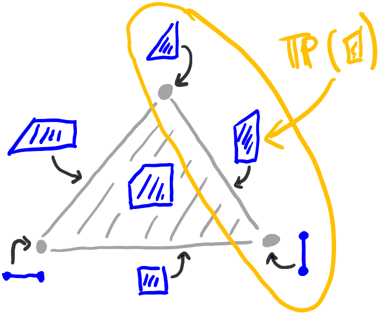
up to translation & dilation

An example: Facet heights

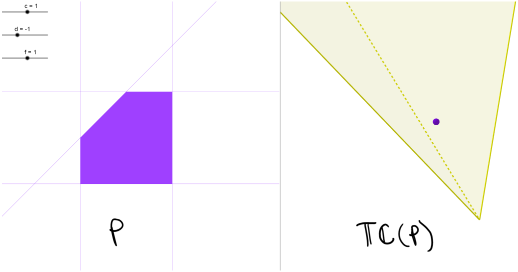


indecomposable

Face poset of TP(P)



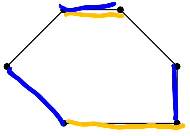
An example: Facet heights



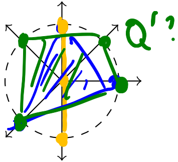
Type cones of polygons

$\mathcal{N}(P)$ = set of unit facet normals of P

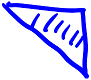
$\bar{Q} + Q' =$



P



$\mathcal{N}(P)$



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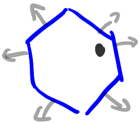
Prop (CDGRY) P = polygon

Faces of $\mathrm{TC}(P)$ correspond to $S \subseteq \mathcal{N}(P)$ s.t. $0 \in \mathrm{relint}(\mathrm{conv}\, S)$

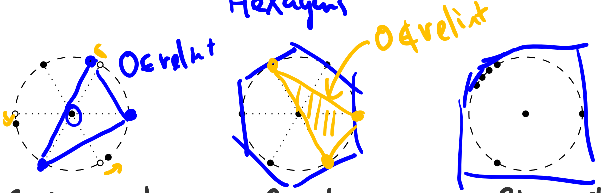
Cor: Any d-polytope w/ $d+3$ vertices is the $\mathrm{TC}(P)$ for some polygon P.

Type cones of polygons

Fix n, consider all n-gons. Which has the "most deformations"? (Maximize f-vector of $\Pi\mathbb{P}$(n-gon))



Hexagons



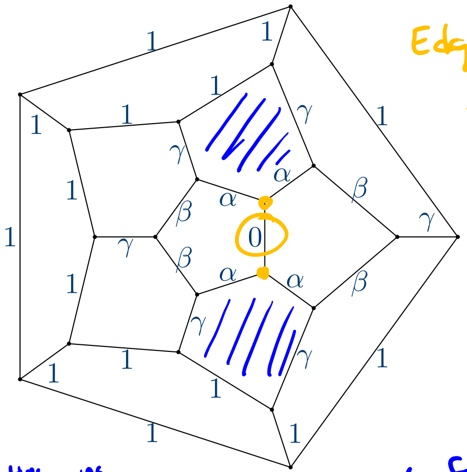
Perturb regular   Regular   Pizza slice

For $n \geq 6$, perturbed regular maximizes f-vector of $\Pi\mathbb{P}$(n-gon)

Example: The dodecahedron

Platonic solids?
(3-polytopes)

Edge weights



facet of
TP(dodec)

f-vector of TP(dodec)

vertices edges ✓ facets

$(278, 2340, 6616, 8812, 6105, 2192, 375, 30)$ ← know all facets!

know exactly 6 of these!

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Type cones of products of simplices

Simplex = trivial type cone

Theorem (McMullen 1973)

Let P be a polytope, $\mathcal{A} = \{a_1, \cdots, a_m\}$ be the vertex set of its polar P°, and $\mathrm{Gale}(A) = \{b_1, \cdots, b_m\}$ be a Gale transform for $\mathcal{A}$. Then

$$\mathbb{TP}(P) \cong \bigcap_{S} \mathrm{conv}\{b_i : b_i \in S\},$$

where the intersection is over all cofacets S of $\mathcal{A}$.

$P \rightsquigarrow P^\circ \rightsquigarrow$ Gale $\rightsquigarrow$ intersect to get faces of $\mathbb{TP}(P)$

Our main result

$\mathbb{TP}(\square) \neq \mathbb{TP}(\pentagon) \neq \mathbb{TP}(\square)$

Theorem (CDGRY)

If P is ***combinatorially isomorphic*** *to a product of $k+1$ nontrivial simplices, $\mathbb{TP}(P)$ is a simplex of dimension k.* ! Simplicial type cone!

In particular, the type polytope of ***any*** *combinatorial d-cube is a $(d-1)$-simplex.*

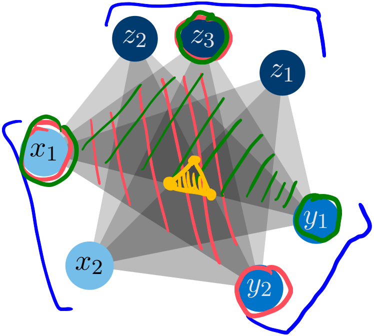
- Only (?) type cone result that is up to combinatorial isomorphism!
- Mirrors AKN result on realization spaces of cubes

"Proof"

Q: Product of simplicial polytopes?

Single simplex $\rightsquigarrow$ polygon?

Key step of proof: Show that the intersection of all **rainbow simplices** from a particular **rainbow configuration** is itself a simplex. !



This rainbow configuration is the Gale transform of the polar of the product of nontrivial simplices. We then apply McMullen's result.

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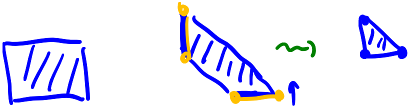
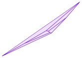
The end



Q:

Thanks for listening!

$\mathbb{TP}($  $) \cong \mathbb{TP}($  $) \cong \mathbb{TP}($  $)$

klee-minty