FOUNDATIONS OF MATROIDS

Matt Baker

joint work with Oliver Lorscheid

the talk was delivered at Fields Matroid Seminar by Matt Baker.
Ahmed Ashraf scribed these notes from the talk.

What is a matroid?

A matroid is a finite set E and non-empty collection $\mathcal{B}$ of subsets such that

$$\forall\, B, B' \in \mathcal{B} \text{ s.t } \forall\, b \in B \setminus B'$$
$$\exists\, b' \in B' \setminus B \text{ s.t } B \cup b' \setminus b \in \mathcal{B}$$
$$\text{and } \quad B' \cup b \setminus b' \in \mathcal{B}$$

← symmetric exchange property

apriori stronger (but actually equivalent)

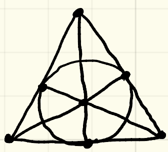
$$\operatorname{rk} M \overset{\text{def}}{=} |B| \qquad \forall\, B \in \mathcal{B}$$

Eg: Let $\mathbb{k}$ be a field.
A $m \times n$ matrix over $\mathbb{k}$.
E columns of A.
$\mathcal{B}$ maximum linearly independent sets of columns of A.

such a matroid is called representable over $\mathbb{k}$

Recall the example of Fano matroid

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} / \mathbb{F}_2 .$$



$M[A]$ depends only on the row space of A.

$$M[A] = M[\operatorname{row}(A)]$$

determined by r-dim'l subspace $W \subseteq \mathbb{k}^n$ where $n = |E|$

i.e. by a pt of $Gr_{\mathbb{k}}(r, n)$

Plücker coordinates $\Delta_I = \det A_I$

where $I \in \binom{E}{r}$ where A is $r \times n$ matrix

$$A \longmapsto (\Delta_I) \in \mathbb{P}^{N-1}$$

representation over $\mathbb{k}$ of matroid over n elements of rank where $N = \binom{n}{r}$

also in Baker-Jin '22

but actually folklore.

Theorem (Folklore) ← Orthogonal matroids

A point $\Delta = (\Delta_I)_{I \in \binom{E}{r}} \in \mathbb{P}^{N-1}$

comes from a subspace W iff $\mathrm{supp}(\Delta_I) = \{\text{basis of a matroid}\}$

and Δ satisfies the 3-term Plücker relations

$$X_{J_{ab}} X_{J_{cd}} - X_{J_{ac}} X_{J_{bd}} + X_{J_{ad}} X_{J_{bc}} = 0$$

3-term Plücker relations generate all Plücker relations.

A **representation** of $\overset{\text{a matroid}}{M}$ over $\mathbb{k}$ is a point $\Delta \in Gr(r,n)(\mathbb{k}) \subseteq \mathbb{P}^{N-1}(\mathbb{k})$ s.t. $\operatorname{supp}(\Delta) = M$.

Two representations Δ, Δ' are **equivalent** if $\exists\, c : E \to \mathbb{k}^{\times}$ s.t. $c : E \longrightarrow \mathbb{k}^{\times}$ s.t.

$$\forall\, I \in \binom{E}{r}, \quad \Delta_I = \Delta'_I \prod_{i \in I} c(i)$$

so Δ and Δ' are in the some orbit of $T = (\mathbb{k}^{\times})^N / \mathbb{k}^{\times}$ on $Gr(r,n)(\mathbb{k})$

We are interested in equivalence classes of representations over $\mathbb{k}$.

Functor:

$$\begin{tikzcd} \text{Fields} \arrow[r] \arrow[d, hook] & \text{Equivalence Classes of representation of } M \\ \text{Pasture} \arrow[ur] & \end{tikzcd}$$

Theorem:

$$\text{Pastures} \longrightarrow \text{Sets}$$

$$\mathbb{K} \longmapsto \text{equivalence classes of representations}$$

is representable by a pasture called the foundation of the matroid M.

For partial fields : Similar theory : Universal Partial Field
Pendavingh – van Zwam

matroids exist
s.t. it is not
representable over
any partial field.

but foundation always exist.
matroids always representable
over some pasture

how do we compute it ?

(student)
Dress-Wenzel

Foundation
of a matroid has a multiplicative group : inner Tutte group.

does it depend
on the field ?

We don't require the additive structure in $\mathbb{K}$ comes from a binary operation.

Eg: $\mathbb{K}$: Krasner hyperfield

$\mathbb{K} = \{0,1\}$ with usual multiplication

$$\left.\begin{array}{l} 0+0=0 \\ 0+1=1 \end{array}\right\} \quad 0+x=0$$

true
in all
pastures

$$1+1=0$$
$$1+1=1$$

example of a pasture.

Eg: $\mathbb{F}_1^{\pm}$ regular partial field

$$\mathbb{F}_1^{\pm} = \{0, 1, -1\} \quad \text{with usual multiplication}$$

$$0 + x = x$$
$$1 + (-1) = 0$$

Pasture: A multiplicatively written monoid F with $0, 1$ s.t. $F^\times = F \setminus \{0\}$ is an abelian gp

$$0 \cdot x = 0 \quad \forall x$$

together with an involution $x \longmapsto -x$ and a set

null $N_F \subseteq \mathrm{Sym}^3(F)$.

Plücker.

secret $(x, y, z) \in N_F$

$x + y + z = 0$

iff $x + y = -z$

① $(0,0,0) \in N_F$ and $(1,0,0) \notin N_F$

② $c \in F^\times$, $(x, y, z) \in N_F$ iff $(cx, cy, cz) \in N_F$

③ $(0, x, y) \in N_F$ iff $x = -y$

generally Tracts

Eg: ① Every field is a pasture.

$(x, y, z) \in N_F$ iff $x + y + z = 0$

② Hyperfields are pastures $\mathbb{K}$ field, $G \leq \mathbb{K}^+$ the multiplicative group

$F = K/G$ is a pasture

$(x,y,z) \in N_F$ iff $\exists\ a,b,c \in G$ s.t
$ax + by + cz = 0$ in $\mathbb{k}$.

Eg: Krasner hyperfield

$$\mathbb{K} = \mathbb{R}/\mathbb{R}^\times = \{0,1\}$$

Eg: $S = \mathbb{R}/\mathbb{R}_{>0}$ ← sign hyperfield

← $1+1+(-1) = 0$
but $1+1+1 \notin N_S$

Partial fields

Let R be a commutative ring. $-1 \in G \leq R^\times$

let $F \overset{\text{be}}{=} G \cup \{0\}$ Then F is a pasture

$x + y + z = 0$ in F iff $= 0$ in R

Eg: $\mathbb{F}_1^\pm$ which can be obtained from $\{1,-1\} \leq \mathbb{Z}$.

Category: Pastures & morphism $\phi : F_1 \longrightarrow F_2$ mult. map
preserving $0, 1, -1$ and $N_{F_1} \to N_{F_2}$

This category has lim, colim, initial $= \mathbb{F}_1^\pm$, final $= \mathbb{K}$
prod, tensor prod

↑ minimal rel. ↑ all relation.

Def: A representation of M over a pasture F is
a pt $\Delta \in \mathbb{P}^{N-1}(\mathbb{F})$

s.t $\text{supp}(\Delta) = M$ and Δ satisfies the
3-term relation

Q: How to deal with associativity of addition.

Thm: (B-Lorscheid) The functor

$$\text{Pastures} \longrightarrow \text{Sets}$$
$$F \longmapsto \{F\text{-rep of } M\}/\sim$$

is representable.

$$\left\{ \text{rep. of } M \atop \text{over } F \right\} / \sim \; = \; \mathrm{Hom}(F_M, F)$$

pasture

Pasture.

original proof used homology theorem

Tutte's theorem: M is regular iff M is rep over GF(2) & GF(3) iff M is rep over every field.

regular matroid theorem

representable over $\mathbb{Z}$

$\mathbb{F}_2 \overset{\text{den}}{=} GF(2)$
$\mathbb{F}_3 \overset{\text{den}}{=} GF(3)$

Observation.

Proof: 1. M is regular iff M is representable over $\mathbb{F}_1^{\pm}$ regular partial field

initial object

in pastures

explicit formula

2. $\mathbb{F}_2 \times \mathbb{F}_3 \cong \mathbb{F}_1^{\pm}$ as pastures.

categorical

3. $\mathrm{Hom}(\mathbb{F}_M, \mathbb{F}_2 \times \mathbb{F}_3) = \mathrm{Hom}(\mathbb{F}_M, \mathbb{F}_2) \times \mathrm{Hom}(\mathbb{F}_M \times \mathbb{F}_3)$

4. $\mathbb{F}_1^{\pm}$ initial in pastures.

contrast with modern proof by Kalman

related to a thm of Tutte

Thm: M is binary (representable over $\mathbb{F}_2$)

iff $\mathbb{F}_M \cong \mathbb{F}_1^{\pm}$ or $\mathbb{F}_2$ ← how Uniform matroid.

Thm: (B-Lorscheid) M is ternary (representable over $\mathbb{F}_3$)

iff $F_M \cong P_1 \otimes \dots \otimes P_k$ where $P_i \in \{\mathbb{U}, \mathbb{D}, \mathbb{H}, \mathbb{F}_3\}$

Notation: If R is commutative ring

$P(R) = PF(G)$. $G = R^{\times} \leq R$ and consider R/G. with $x+y+z=0$ if $x+y+z=0 \in R$

1. $P(\mathbb{Z}) = \mathbb{F}_1^{\pm}$

2. $P(\mathbb{Z}[\frac{1}{2}]) = \mathbb{D}$

3. $P(\mathbb{Z}[\zeta_6]) = \mathbb{H}$

4. $P(\mathbb{Z}[T, \frac{1}{T}, \frac{1}{1-T}]) = \mathbb{U}$

cross ratio in projective geometries

Study of cross ratios in matroid theory

4 pts on a projective line have a cross ratio. $\longleftrightarrow$ $U(2,4)$

Can give a presentation for F_M in terms of cross ratio

One uses Tutte's homotopy theorem.

Application: Suppose we take $\mathbb{F}_3$ & $\mathbb{S} = \mathbb{R}/\mathbb{R}_{>0}$ (sign hyperfield)

repr. of M over $\mathbb{S}$ iff $\exists$ orientation of M.

Theorem (Lee-Scobie) ← their proof is hammering

M is rep'ble over $\mathbb{F}_3$ & $\mathbb{S}$ iff M rep'ble over $\mathbb{D}$.

<u>Proof</u>: (Sketch)

$$\mathbb{D} \longrightarrow \mathbb{F}_3 \times \mathbb{S}$$

deg 2 homomorphism

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "\text{hom}"] & & \mathbb{F}_3 \times \mathbb{S} \\ & F_M = \bigotimes P_i \arrow[ul, "\exists!"] \arrow[ur] & \end{tikzcd}$$

$\mathbb{D}, \mathbb{U}, \mathbb{H}, \mathbb{F}_3$

coreflections?

Q: Is there a converse question.

symmetric exchange property $\Longleftrightarrow$ Plücker relations over $\mathbb{K}$

final field

Krasner field

Christian Haase   If I hand you a pasture whether it is a foundation

$$\begin{tikzcd} \mathbb{F}_2 \arrow[dr] & \\ & \mathbb{F}_1 \otimes \mathbb{F}_2 \\ \mathbb{F}_1 \arrow[ur] & \end{tikzcd}$$

Fields don't generally have tensor product.

Yannic Vargas

Are there geometrical properties of the matroid polytope coming from the representability of the functor characterizing M.

Chris Eur $\longrightarrow$ Hodge theory

Chern class

generators
presentation

Tutte group.

$F_M^{\times}$ = inner Tutte group

$F_M \longrightarrow$

odd pasture $1 \neq -1$

$\mathbb{F}_2 \times \mathbb{F}$