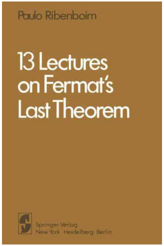
Primes, postdocs and pretentiousness

Andrew Granville

Université de Montréal

Fields-PIMS-CRM prize lecture
20th October 2022



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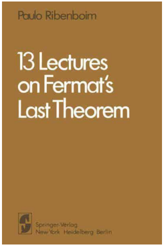

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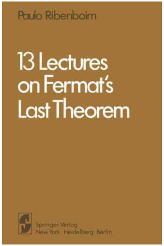

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Faltings' 1983 Theorem – proof of Mordell's conjecture

Any curve, defined in $\mathbb{Q}$, of genus > 1, contains only finitely many rational points. This includes $u^n + v^n = 1$ when $n > 3$

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Corollary to Faltings' Theorem

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Therefore if prime $p > 3$ divides n then $n = pm \leq b_p := p \log_2 B_p$.

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"Easy" sieve result: A proportion $1 - \epsilon_y$ of the integers have a prime factor in $[5, y]$, where $\epsilon_y \to 0$ as $y \to \infty$.

This implies FLT is true for 100% of exponents n. □

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AG-Monagan (1988)

The first case of Fermat's last theorem is true for all prime exponents up to $714,591,416,091,389$.

We showed: FLTI false implies $q^{p-1} \equiv 1 \pmod{p^2}$ for all $q \leq 89$.

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so $c \leq \kappa^7 \implies$ **Bounded number of FLT solns with $n \geq 4$.**

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- $x^p + y^q = z^n$ with $(x, y) = 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{n} < 1$;
- $F(x, y) = z^n$ with $(x, y) = 1$ and $2/d + 1/n < 1$ where $F(\cdot, \cdot)$ is a binary form of degree d.

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Generalization: Let $F(x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots, x_m]$. Are there integer solutions $\ell_1, \ldots, \ell_m$ to

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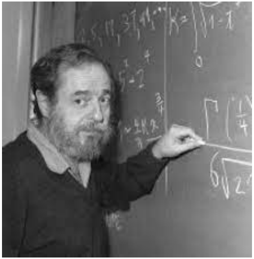
AG: If there are no solutions to $F(\zeta_1, \ldots, \zeta_m) = 0$ in roots of unity, then $\exists$ integer solns to (1) for very "few" exponents n.

Gauss's letter to Sophie Germain, 1807

"A taste for the abstract sciences in general and above all the mysteries of numbers is excessively rare. One is not astonished by it for the enchanting charms of this sublime science are revealed only to those who have the courage to go deeply into it. However, when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents and superior genius. Indeed nothing could prove to me in so flattering and unequivocal manner that the attractions of this science, which have enriched my life with so many joys, are not illusory, than the attention with which you have honored it."

Postdoc at Toronto, 1987-89 with John Friedlander



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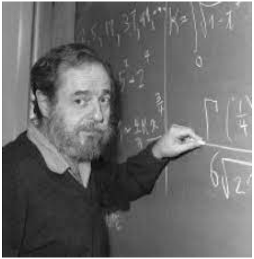

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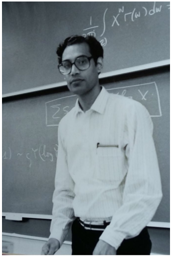

Kumar Murty

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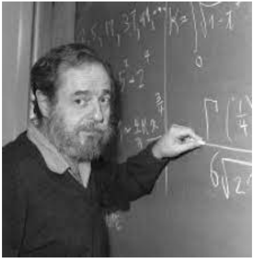

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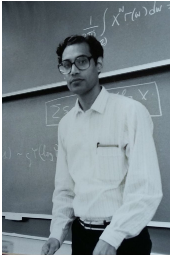

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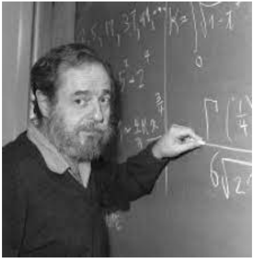

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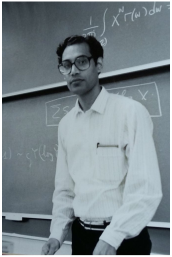

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If $(a, q) = 1$ then

$$\pi(x; q, a) = \#\{ \text{ primes } p \le x : p \equiv a \pmod q\} \sim \frac{\pi(x)}{\phi(q)}$$

where $\pi(x) = \#\{\text{primes } p \le x\}$ and
$\phi(q) = \#\{a \in [1, q] : \ (a, q) = 1\}$.

(Prime Number Theorem for Arithmetic Progressions – PNT4APs)

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Friedlander-AG, 1989 – disproof of Elliott-Halberstam conj

(2) does not hold for "almost all" q with $x = q(\log q)^A$, for some $(a, q) = 1$.

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PNT4APs:
$$\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)} \tag{2}$$

holds for $x \geq e^{q^\epsilon}$. And for $x \geq q^2(\log q)^{2+\epsilon}$ assuming GRH.

Bombieri-Vinogradov Theorem ($\approx$ 1965)

(2) holds for "almost all" $x \geq q^2(\log q)^{1+\epsilon}$, for all $(a, q) = 1$.

Exponent "2" a barrier to progress. Can exponent "1" hold always?

Friedlander-AG, 1989 – disproof of Elliott-Halberstam conj

(2) does not hold for "almost all" q with $x = q(\log q)^A$, for some $(a, q) = 1$.

Elliott-Halberstam conj, II: (2) holds for "almost all" q, for all $x \geq q^{1+\epsilon}$, for all $(a, q) = 1$.

Postdoc at IAS Princeton, 1989-91 with Enrico Bombieri

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Atle Selberg

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Riemann Hypotheses (GRH), 1859+

- $L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$ for $\text{Re}(s) > 1$, with $\chi(\cdot)$ a character.

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- Analytically continue it to all of $\mathbb{C}$ (except perhaps at $s = 1$).
- Guess: If $L(\rho, \chi) = 0$ with $0 < \text{Re}(\rho) < 1$ then $\text{Re}(\rho) = \frac{1}{2}$.

A more moderate ambition than the Riemann Hypothesis

Let $\chi(\cdot)$ be a Dirichlet character mod q. Define for $\operatorname{Re}(s) > 1$

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Life goal – Prove there are no "Siegel zeros"!

($L(\beta, \chi) \neq 0$ whenever $\beta > 1 - \frac{1}{\log q}$ for real characters χ)

Proving there are no Siegel zeros

AG-Stark, 2000

If the "uniform" *abc*-conjecture holds in "Hilbert class fields" then there are no Siegel zeros for $L(s, (\frac{D}{\cdot}))$ where $D < 0$.

That is, if $L(\beta, (\frac{D}{\cdot})) = 0$ with $\beta \in \mathbb{R}$ and $D < 0$ then

$$\beta < 1 - \frac{1}{\log |D|}.$$

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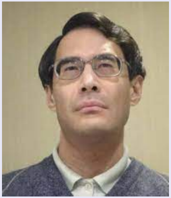
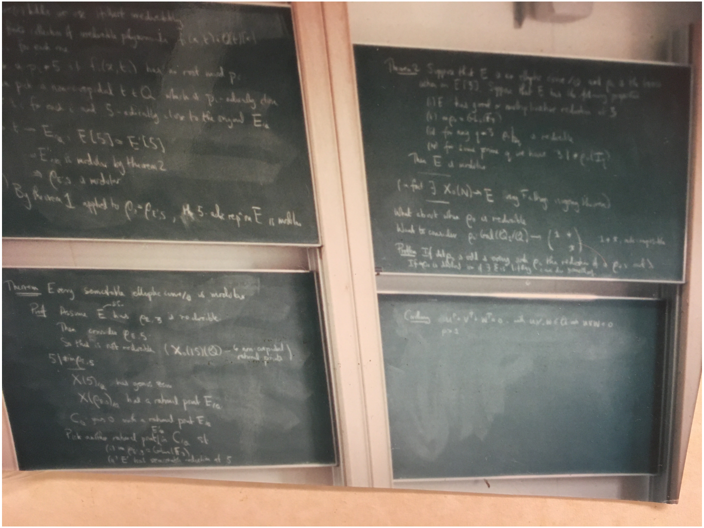
Mochizuki-Fesenko-Hoshi-Minamide-Porowski, Nov 2020



A modification of this version of *abc* can be proved unconditionally!

"Proof" gives bounds on solns to Fermat equation in number fields.

At a meeting at the Isaac Newton Institute, June 23,1993



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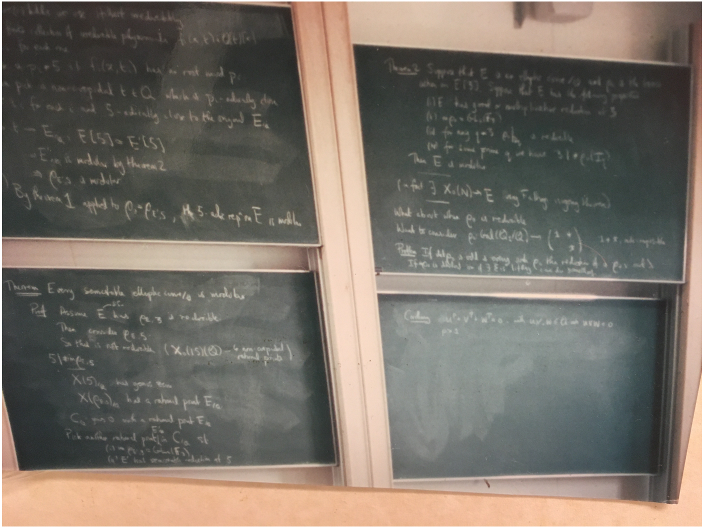

Corollary: $u^p + v^p + w^p = 0$ $(p > 2)$ with $u, v, w \in \mathbb{Q} \implies uvw = 0$.

Back to $x^p + y^q = z^r$

Darmon-AG, 1995

For fixed integers p, q, r with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ there are only finitely many integer solutions to

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A much more subtle Corollary to Faltings' Theorem.

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Perhaps none with $p, q, r \geq 3$?

Univ of Georgia: Students and Postdocs

Anitha Srinivasan

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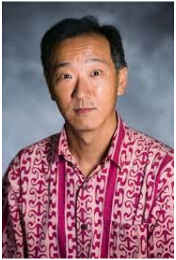

Ken Ono

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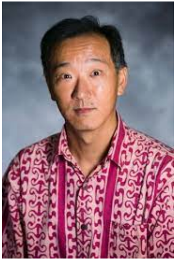

Anitha Srinivasan   Ken Ono   Ernie Croot

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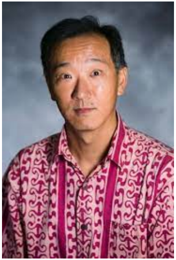

Ken Ono

Ernie Croot

Ernie Croot 2003 – The Erdős-Graham coloring conjecture

There exists a constant $b > 0$ such that if we r-color the integers then there exists a monochromatic subset S of $[2, b^r]$ such that

$$\sum_{n \in S} \frac{1}{n} = 1.$$

Negative mean values of multiplicative functions

K Soundararajan

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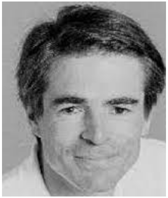
AG-Soundararajan, 2001

The number of -1's is always $\leq \{c + o(1)\}x$ where

$$c = \log(1 + \sqrt{e}) - 2\int_1^{\sqrt{e}} \frac{\log t}{t+1}\, dt = .828499\ldots$$

Attained if $f(p) = 1$ for $p < x^{1/(1+\sqrt{e})}$ and $f(p) = -1$ otherwise.

Université de Montréal (2002–): Primes and pretentions

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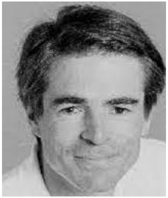   

PRIME PATTERNS

Are there infinitely many prime *twins*, $p, p+2$?

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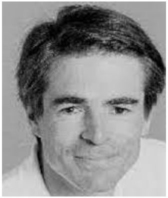


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Celebrity Twin Prime Video

Prime patterns and pretentiousness

$$p, p+2$$

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$$p, p + 2$$

$$p, p + 4$$

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$$p, p+4 \text{ or } p+6 \text{ or } \ldots$$

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$$2p + 1, 4p + 1 \text{ and } 6p + 5$$

$$p, p + d, p + 2d, \ldots, p + kd$$

Any pattern except if obvious reason why not, like $n, n + 1$.

The GPY story, I

Let $p_1 = 2, p_2 = 3, \ldots$ be the sequence of primes.

Wts, Inf many n such that $p_{n+1} - p_n = 2$.

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Up to 2000, best result known $< \frac{1}{4} \log x$.

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Found the mistake in one of those lemmas!:

High dimensional geometry is not like low-dimensional geometry.

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In 2004, Ben Green (postdoc at UBC) came to U de M for a visit. Working on his first project with Terry Tao on prime patterns

$$p, p+d, p+2d, p+3d.$$

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Three days later ...

Ben Green and Terry Tao, 2005

There are infinitely many k-term arithmetic progressions of primes.

The GPY story, III

Dan Goldston, Janos Pintz and Cem Yildirim, 2009

There are infinitely many primes p_n with

$$p_{n+1} - p_n \leq \sqrt{\log p_n}.$$

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There are infinitely many primes p_n with

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Proof uses GPY sieve weights but a *version* of the Bombieri-Vinogradov Thm (with $x \geq q^{5/3}$) Very very tough stuff

The GPY story, IV

James Maynard
(CRM-ISM postdoc 2013-14)

Perhaps we can modify the GPY sieve to obtain Zhang's result, and only use Bombieri-Vinogradov? It would be simpler.

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Also infinitely many primes p_n with

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Very similar results proved at the same time by Terry Tao.

The GPY story, V

And then towards the end of James's year in Montreal:

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We believe if x is suff large then

$$\max_{p_n \le x} p_{n+1} - p_n \ge (\log x)^2.$$

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$$\max_{p_n \leq x} p_{n+1} - p_n \geq c \log x \frac{\log \log x \log \log \log \log x}{(\log \log \log x)^2}$$

Erdős: \$ 10,000 to prove that one can let $c \to \infty$ as $x \to \infty$.

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James Maynard, 2016 + Ford, Green, Konyagn & Tao

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Define

$$\lambda(n) = (-1)^{\Omega(n)} \text{ a multiplicative function.}$$

PNT $\iff$ $\frac{1}{x}\sum_{n\le x}\lambda(n) \to 0$ as $x \to \infty$.

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If $\lambda(n) = (-1)^{\Omega(n)} = 1$ for most n, then $\lambda(2n) = -1$, a contradiction!

Linnik's Theorem (1944)

YURI LINNIK: *There exists a constant L such that any arithmetic progression*

$$a, a+d, a+2d, \ldots$$

with $\gcd(a,d)=1$ *contains a prime* $p = a + nd$ *with* $p \leq d^L$.

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"Repulsion principles": Zeros of polynomials, and of L-functions cannot be close together.

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Koukoulopoulos, 2013 – Strongest known unconditional PNT

$$\left|\pi(x) - \int_2^x \frac{dt}{\log t}\right| \leq c\, x \exp\left(- c' \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)$$

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AG-Harper-Sound, 2019

Explains new theory in 35 pages, including the pretentious large sieve, and proofs of Linnik's Theorem and Hoheisel's Theorem.

Annals of Mathematics **184** (2016), 1–42

Multiplicative functions in short intervals

By Kaisa Matomäki and Maksym Radziwiłł

Dedicated to Andrew Granville

Abstract

We introduce a general result relating "short averages" of a multiplicative function to "long averages" which are well understood. This result has several consequences. First, for the Möbius function we show that there are cancellations in the sum of $\mu(n)$ in almost all intervals of the form $[x, x+\psi(x)]$ with $\psi(x) \to \infty$ arbitrarily slowly. This goes beyond what was

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2019 *New Horizons in Mathematics* Prize

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Paul Erdős and Terry Tao

Let $a_1, a_2, \ldots$ be a sequence of 1's and -1's. The sums

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get arbitrarily big (any d, any N).

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Using Matomäki-Radziwiłł: If such sums stay small then

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The main work has been on their correlations, due to Klurman, Mangerel, Matomäki, Radziwiłł, Shao, Tao, Teräväinen, Ziegler, ...

Noblest courage, extraordinary talents and superior genius